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Exam ASTAM Study Manual

2nd Edition

Abraham Weishaus, Ph.D., FSA, CFA, MAAA

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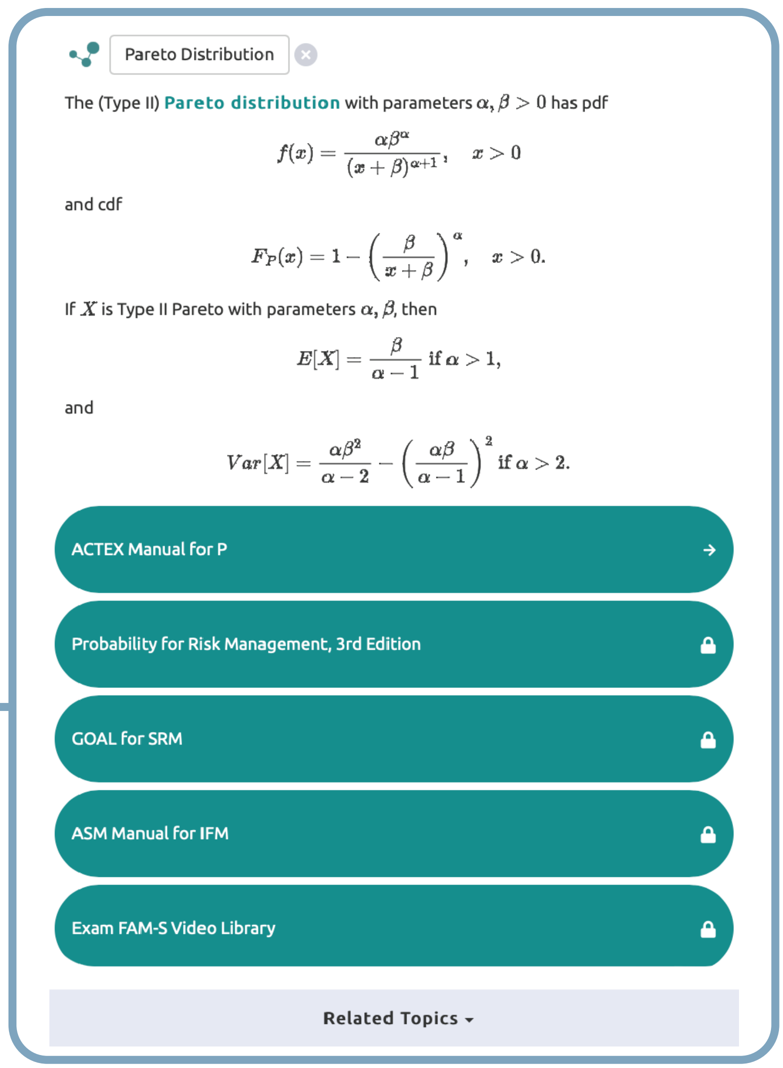

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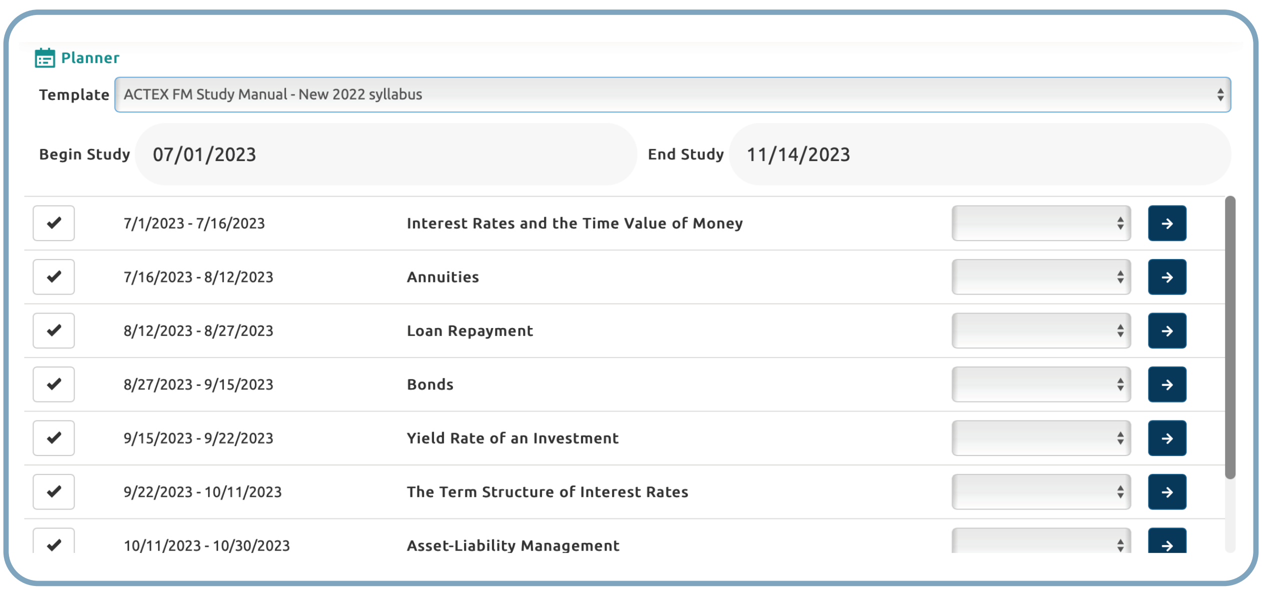

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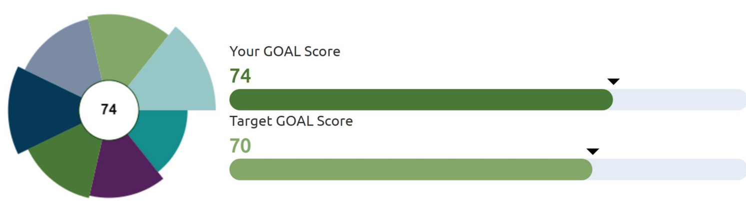

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64 A. Multiple State Models Weighting: 20%	71 B. Multiple Decrement Models Weighting: 10%	73 C. Multiple Life Models Weighting: 10%
82 D. Profit Analysis Weighting: 15%	76 E. Universal Life Insurance Weighting: 15%	78 F. Embedded Options and Equity-Linked Insurance Weighting: 15%
87 G. Pension Plans and Retirement Benefits Weighting: 15%		

See key areas where you can improve.

A. Multiple State Models — A1. Multiple State Probabilities

Difficulty	Mean	Proportion
Core	38	22 / 48
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03/24/2023	03/24/2023	03/24/2023	Quiz	G. Pension Plans and Retirement Benefits	15	Complete	Review
03/24/2023	03/24/2023	03/24/2023	Quiz	C1. Joint Life/Last Survivor Probabilities, C2. Contingent Probabilities, C3. Expectation and Variance, C6. ML Models with Transition Intensities, C4. ML Models: Premiums and Reserves (Discrete), C5. ML Models: Premiums and Reserves (Continuous)	15	Complete	Review
03/24/2023	03/24/2023	03/24/2023	Quiz	F. Embedded Options and Equity-Linked Insurance	20	Complete	Review
03/24/2023	03/24/2023	03/24/2023	Quiz	D. Profit Analysis	10	Complete	Review
09/19/2022	09/19/2022	09/19/2022	Quiz	G4. Pension Plans and Retirement Benefits (Written Answers)	2	Complete	Review

Quickly return to previous sessions.

Contents

Preface

Exam ASTAM is a 3-hour written answer exam. It extends the discussion of mathematics of short-term insurance found in Exam FAM-S. FAM-S is the prerequisite for this course. There may be questions on FAM-S material on the exam. For example, while maximum likelihood estimation is not on the ASTAM syllabus, variance of maximum likelihood estimators is on the syllabus, and any question on variance of MLEs is likely to start off asking you to calculate the maximum likelihood estimator. In fact, released exams have questions asking you to calculate maximum likelihood estimators.

Useful materials

The exercises in this manual

The manual has short answer and multiple-choice exercises. Even though the exam will consist of multi-part written answer questions, these exercises will help you learn the material.

All SOA or joint exam questions in this manual from exams given in 2000 and later, with solutions, are also available on the web from the SOA. When the 2000 syllabus was established in 1999, sample exams 3 and 4 were created, consisting partially of questions from older exams and partially of new questions, not all multiple choice. These sample exams were not real exams, and some questions were inappropriate or defective. These exams are no longer posted on the web. I have included appropriate questions, labeled "1999 C3 Sample" or "1999 C4 Sample". *These refer to these 1999 sample exams, not the current sample questions.*

Questions from old exams are marked xxx:yy, where xxx is the time the exam was given, with S for spring and F for fall followed by a 2-digit year, and yy is the question number. Sometimes xxx is preceded with SOA or CAS to indicate the sponsoring organization. From about 1986 to 2000, SOA exams had 3-digit numbers (like 160) and CAS exams were a number and a letter (like 4B). From 2000 to Spring 2003, exam 3 was jointly sponsored, so I do not indicate "SOA" or "CAS" for exam 3 questions from that period. There was a period in the 1990's when the SOA, while releasing old exam questions, did not indicate which exam they came from. As a result, I sometimes cannot identify the source exam for questions from this period. In such a case, I mark the question aaa-bb-cc:yy, where aaa-bb-cc is the study note number and yy is the question number. Generally aaa is the exam number (like 160), and cc is the 2-digit year the study note was published.

Downloads from the SOA

The most important download is the syllabus. For the Spring 2024 exam, the link is

`https://www.soa.org/4ae244/globalassets/assets/files/edu/2024/spring/syllabi/2024-spring-exam-astam-syllabus.pdf`

The syllabus links to the introductory study note,

`https://www.soa.org/globalassets/assets/files/edu/2024/spring/intro-notes/2024-spring-exam-astam-intro-note.pdf`

The introductory study note links to the two study notes, a notation and terminology note, sample questions and solutions, and a formula sheet. The link for the formula sheet is

`https://www.soa.org/globalassets/assets/files/edu/2023/astam-formula-sheet.pdf`

Note that at the exam you will not be given the distribution tables that you were given for Exam FAM. The formula sheet has a smaller set of distributions than the FAM-S tables. The formula sheet does have many of the more complicated formulas, so you don't have to memorize them. Strangely, the formula sheet does not have the $(a, b, 1)$ class of discrete distributions, yet it has the recursive formula for compound distributions having $(a, b, 1)$ primary distributions. (Lesson 15)

At the exam you will be given a spreadsheet that has all of the Excel functions, but does not have Solver or VBA. You can use this spreadsheet to calculate the normal distribution function as well as anything else you may want to calculate. Many of the methods you'll learn on the exam require extensive calculations, making worksheets very helpful.

Notes About the Exam

Form of the Exam

The exam has 6 written answer questions with a total of 60 points, each question with several parts. You are given 3 hours to complete it.

New for Spring 2024, one of the questions will be a workbook question. You can use the worksheet you are given to carry out calculations for any question, but for this particular question you must carry out all work in the workbook. My guess is that the workbook question will have a relatively large set of data. It is possible that worksheet functions will be necessary to solve the question.

Although this manual does not have workbook questions, the three practice exams have one workbook question apiece.

Topic weights

The syllabus breaks the course down into 7 topics. Their weights and the lessons in the manual that cover them are:

Topic	Weight	Lessons
1. Severity Models	12–22%	1–2, 8
2. Aggregate Models	12–22%	12–13, 15, 17
3. Coverage Modifications	6–12%	3–10, 14, 16
4. Construction and Selection of Parametric Models	12–22%	19–23
5. Credibility	12–20%	25–37
6. Reserving for Short Term Insurance Coverages	12–20%	39–43
7. Pricing for Short-Term Insurance Coverages	4–12%	44

Study Schedule

Different students will have different speeds and different constraints, so it's hard to create a study schedule useful for everybody. However, here is a sample 11-week study schedule:

Week	Subject	Lessons	Rarely Tested
1	Parametric distributions	1–2,8	1.4
2	Coverage modifications	3–6	
3	Coverage modifications	7–10	*9*
4	Aggregate losses	12–17	
5	Maximum likelihood	19	*19.2.3*,*19.4.2*
6	Selecting and fitting models	20–23	
7	Bayesian credibility	25–29	*29.2*
8	Bühlmann credibility	30–32	
9	Bühlmann credibility	33–37	33.2,35.1,35.3,37.2
10	Loss reserving	39–43	
11	Ratemaking	44	

The last column lists rarely tested topics so you can skip those if you are behind in your schedule. Italicized sections in this column are, in my opinion, extremely unlikely exam topics. However, it is possible that with the new written-answer format of the exam some topics rarely tested in the past will be tested more frequently.

Errata

Please report any errors you find. Reports may be sent to the publisher (mail@studymanuals.com) or directly to me (errata@aceyourexams.net). *When reporting errata, please indicate which manual and which edition and printing you are referring to!* This manual is the second edition of the Exam ASTAM manual.

An errata list will be posted at http://errata.aceyourexams.net

Acknowledgements

I wish to thank the Society of Actuaries and the Casualty Actuarial Society for permission to use their old exam questions. These questions are the backbone of this manual.

I wish to thank Donald Knuth, the creator of TEX, Leslie Lamport, the creator of LATEX, and the many package writers and maintainers, for providing a typesetting system that allows such beautiful typesetting of mathematics and figures.

Part I

Parametric Distributions

Lesson 2

Mixtures and Splices

Reading: *Loss Models* Fifth Edition 5.2.4–5.2.6

Mixtures are on the FAM syllabus as well as the ASTAM syllabus; if you are comfortable with them, you may just scan Subsections 2.1.1–2.1.2 quickly. But frailty models and splices are not on the FAM syllabus, so make sure to study those parts of Section 2.1.

2.1 Mixtures

2.1.1 Discrete mixtures

A (finite) **mixture distribution** is a random variable X whose distribution function can be expressed as a weighted average of n distribution functions of random variables X_i, $i = 1, \ldots, n$. In other words,

$$F_X(x) = \sum_{i=1}^{n} w_i F_{X_i}(x)$$

with the weights $w_i \geq 0$ adding up to 1. Since the density function is the derivative of the distribution function, the density function is the same weighted average of the individual density functions:

$$f_X(x) = \sum_{i=1}^{n} w_i f_{X_i}(x)$$

If discrete variables are mixed, the probabilities of the mixture are the weighted averages of the component probabilities.

For example, suppose X is a mixture of an **exponential distribution** with mean 100 and weight 60% and an exponential distribution with mean 200 and weight 40%. Then the probability that $X \leq 100$ is

$$\Pr(X \leq 100) = 0.6(1 - e^{-100/100}) + 0.4(1 - e^{-100/200}) = 0.6(0.6321) + 0.4(0.3935) = \boxed{0.5367}$$

A mixture is not the same as a sum of random variables! The distribution function for a sum of random variables—even when they are identically distributed—is usually difficult to calculate. It is important not to confuse the two. Let's consider the situations where each would be appropriate.

A **sum of random variables** is an appropriate model for a situation where several distinct events occur, and you are interested in the sum. Each event may have the same distribution, or may not. Examples of sums of random variables are:

1. The total of a random sample of n items is a sum of random variables. For a random sample, the items are independent and identically distributed.

2. Aggregate loss on a policy with multiple coverages is a sum of random variables, one for each coverage. If a homeowner's policy has coverage for fire, windstorm, and theft, the aggregate loss for each of these three coverages could have its own distribution X_i, and then the aggregate loss for the entire policy would be $X_1 + X_2 + X_3$. These distributions would be different, and may or may not be independent.

A mixture distribution is an appropriate model for a situation where a single event occurs. However, the single event may be of many different types, and the type is random. For example, let X be the cost of a dental claim. This is a single claim and X has a distribution function. However, this claim could be for **preventative work** (cleaning and scaling), **basic services** (fillings), or **major services** (crowns). Each type of work has a distribution X_i. If 40% of the claims are for preventative work, 35% for basic services, and 25% for major services, then the distribution of X will be a weighted average of the distributions of the 3 X_i's: $F_X(x) = 0.4F_{X_1}(x_1) + 0.35F_{X_2}(x_2) + 0.25F_{X_3}(x_3)$. It is *not* true that $X = 0.4X_1 + 0.35X_2 + 0.25X_3$. In fact, the type of work that occurred is random; it is not true that every claim is 40% preventative, 35% basic, and 25% major. If that were true, there would be less variance in claim size!

Since a mixture is a single random variable, it can be used as a model even when there is no justification as given in the last paragraph, if it fits the data well.

For calculating means, the **mean of a mixture** is the weighted average of the means of the components. Since the densities are weighted averages and the expected values are integrals of densities, the expected value of a mixture is the weighted average of the expected values of the components. This is true for any raw moment, not just the first moment. But this immediately implies that the **variance of a mixture** is *not* the weighted average of the variances of its components. You must compute the variance by computing the second moment and then subtracting the square of the mean.

Example 2A Losses on an auto liability coverage follow a distribution that is a mixture of two Paretos. Each distribution in the mixture has equal weight. One distribution has parameters $\alpha = 3$ and $\theta = 1000$, and the other has parameters $\alpha = 3$ and $\theta = 10{,}000$.

Calculate the variance of a loss. ■

Solution: Let X be loss size. We have

$$\mathbf{E}[X] = 0.5\left(\frac{1000}{2}\right) + 0.5\left(\frac{10{,}000}{2}\right) = 2750$$

$$\mathbf{E}[X^2] = 0.5\left(1000^2\right) + 0.5\left(10{,}000^2\right) = 50{,}500{,}000$$

$$\operatorname{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2 = 50{,}500{,}000 - 2750^2 = \boxed{42{,}937{,}500}$$ □

Example 2B The severity distribution for losses on an auto collision coverage is as follows:

$$F(x) = 1 - 0.7\left(\frac{2000}{2000+x}\right)^3 - 0.3\left(\frac{7500}{7500+x}\right)^4 \qquad x \geq 0$$

Calculate the coefficient of variation of loss size. ■

Solution: This distribution is a mixture of two Pareto distributions, the first with parameters $\alpha = 3$, $\theta = 2000$ and the second with $\alpha = 4$, $\theta = 7500$. We calculate the first two moments:

$$\mathbf{E}[X] = 0.7\left(\frac{2000}{2}\right) + 0.3\left(\frac{7500}{3}\right) = 1450$$

$$\mathbf{E}[X^2] = 0.7\left(\frac{2(2000)^2}{(2)(1)}\right) + 0.3\left(\frac{2(7500^2)}{(3)(2)}\right) = 8{,}425{,}000$$

$$\operatorname{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2 = 8{,}425{,}000 - 1450^2 = 6{,}322{,}500$$

It then follows that the coefficient of variation is

$$\mathrm{CV} = \frac{\sqrt{6{,}322{,}500}}{1450} = \boxed{1.7341}$$ □

Example 2C On an auto collision coverage, there are two classes of policyholders, A and B. 70% of drivers are in class A and 30% in class B. The means and variances of losses for the drivers are:

Class	Mean	Variance
A	300	30,000
B	800	50,000

A claim is submitted by a randomly selected driver.

Calculate the variance of the size of the claim. ■

Solution: This is a mixture situation—a single claim, with probabilities of being one type or another. Let X be claim size.

$$\mathbf{E}[X] = 0.7(300) + 0.3(800) = 450$$
$$\mathbf{E}[X^2] = 0.7(30{,}000 + 300^2) + 0.3(50{,}000 + 800^2) = 291{,}000$$
$$\mathrm{Var}(X) = 291{,}000 - 450^2 = \boxed{\mathbf{88{,}500}}$$ □

2.1.2 Continuous mixtures

So far we've discussed discrete mixtures. It is also possible for mixtures to be **continuous**. This means that the distribution function of the mixture is an integral of parametric distribution functions of random variables, and a parameter varies according to a distribution function. The latter distribution is called a *mixing distribution*. Here is an example.

Example 2D The number of losses on a homeowner's policy is binomially distributed with parameters $m = 5$ and q. The parameter q varies by policyholder uniformly between 0 and 0.4.

Calculate the probability of 2 or more losses for a policyholder. ■

Solution: For a single policyholder,

$$\Pr(X = 0 \mid q) = (1-q)^5$$
$$\Pr(X = 1 \mid q) = 5q(1-q)^4$$

To calculate the probability for a randomly selected policyholder, we integrate over q using the uniform density function, which here is $\frac{1}{0.4}$, as the weight.

$$\begin{aligned}\Pr(X=0) &= \frac{1}{0.4}\int_0^{0.4}(1-q)^5\,\mathrm{d}q\\ &= -\frac{1}{0.4}\frac{(1-q)^6}{6}\bigg|_0^{0.4}\\ &= \frac{1}{2.4}(1-0.6^6)\\ &= \tfrac{5}{12}(1-0.6^6)\\ \Pr(X=1) &= \frac{5}{0.4}\int_0^{0.4} q(1-q)^4\,\mathrm{d}q\end{aligned}$$

This is easier to integrate by substituting $u = 1 - q$.

$$\begin{aligned}
&= \frac{5}{0.4}\int_{0.6}^{1}(1-u)u^4\mathrm{d}u\\
&= 12.5\left(\frac{u^5}{5}-\frac{u^6}{6}\right)\Bigg|_{0.6}^{1}\\
&= 12.5\left(\frac{1-0.6^5}{5}\right) - 12.5\left(\frac{1-0.6^6}{6}\right)\\
&= 2.5(1-0.6^5) - \tfrac{25}{12}(1-0.6^6)\\
\Pr(X=0)+\Pr(X=1) &= 2.5(1-0.6^5) - \tfrac{20}{12}(1-0.6^6)\\
&= 2.3056 - 1.5889 = 0.7167
\end{aligned}$$

Therefore, the probability of 2 or more losses is the complement of 0.7167, which is $1 - 0.7167 =$ **0.2833**. □

2.1.3 Frailty models

A special type of continuous mixture is a ***frailty model***. These models can be used to model loss sizes or survival times. However, the following discusses frailty models only in the context of survival times.

Suppose the **hazard rate** for each individual is $h(x \mid \Lambda) = \Lambda a(x)$, where $a(x)$ is some continuous function and the multiplier Λ varies by individual. Thus the shape of the hazard rate function curve does not vary by individual. If you are given that A's hazard rate is twice B's at time 1, that implies Λ for A is twice Λ for B. That in turn implies that A's hazard rate is twice B's hazard rate *at all times*.

Assume that $h(x) = 0$ for $x < 0$. Then the **cumulative hazard rate function** is $H(x) = \int_0^x h(t)\mathrm{d}t$, and the **survival function** can be expressed as

$$S(x) = e^{-H(x)}$$

Now let $A(x) = \int_0^x a(t)\mathrm{d}t$. Then $H(x \mid \Lambda) = \int_0^x \Lambda a(t)\mathrm{d}t = \Lambda A(x)$ and

$$S(x \mid \Lambda) = e^{-H(x\mid\Lambda)} = e^{-\Lambda A(x)} \qquad (*)$$

By definition, $S(x) = \Pr(X > x)$, so by the **Law of Total Probability**,

$$S(x) = \Pr(X > x) = \int_0^\infty \Pr(X > x \mid \lambda) f(\lambda)\mathrm{d}\lambda = \mathbf{E}_\Lambda\big[\Pr(X > x \mid \Lambda)\big] = \mathbf{E}\big[S(x \mid \Lambda)\big]$$

Plugging in $S(x \mid \Lambda)$ from $(*)$, the unconditional or marginal survival rate $S(x)$ is

$$S_X(x) = \mathbf{E}_\Lambda\big[S(x \mid \Lambda)\big] = \mathbf{E}_\Lambda\left[e^{-\Lambda A(x)}\right] = M_\Lambda\big(-A(x)\big) \qquad (2.1)$$

where $M(x)$ is the **moment generating function**.

In a frailty model, typical choices for the conditional hazard rate given Λ are:

- Constant hazard rate, or exponential. This can be arranged by setting $a(x) = 1$ (or $a(x) = k$ for any constant k).
- **Weibull**, which can be arranged by setting $a(x) = \gamma x^{\gamma-1}$

Typical choices for the distribution of Λ are **gamma** and **inverse Gaussian**, some of the few distributions for which the moment generating function has a closed form.

Frailty models rarely appear on exams. If they do appear, I would not expect it to be labeled as a "frailty model", nor would I expect the specific notation (such as $a(x)$) to be used. Instead, you would be given an appropriate hazard rate conditional on a parameter and a distribution for the parameter.

EXAMPLE 2E For a population following a frailty model, you are given

(i) $a(x) = 1$

(ii) Λ has a gamma distribution with $\alpha = 0.2$ and $\theta = 0.1$.

For a randomly selected individual from the population:

1. Calculate the probability of surviving to 70.
2. Calculate mean survival time and the variance of survival time.

SOLUTION: 1. Use of $a(x) = 1$ leads to an exponential model. We have

$$A(x) = \int_0^x 1 \mathrm{d}t = x$$
$$S(x \mid \Lambda) = e^{-\Lambda A(x)} = e^{-\Lambda x}$$

The moment generating function for a gamma (which appears in the formula sheet) is $M(t) = (1 - \theta t)^{-\alpha}$. In this example, $M_\Lambda(t) = 1/\sqrt[5]{1 - 0.1t}$. Then

$$\begin{aligned} S(70) &= M_\Lambda(-70) \\ &= \frac{1}{\sqrt[5]{1 - 0.1(-70)}} \\ &= \frac{1}{\sqrt[5]{8}} = \boxed{0.659754} \end{aligned}$$

2. By equation (2.1), the survival function is

$$S(x) = M(-x) = \left(\frac{1}{1 + 0.1x}\right)^{0.2} = \left(\frac{10}{10 + x}\right)^{0.2}$$

which we recognize as a two-parameter Pareto with $\alpha = 0.2$, $\theta = 10$. For such a distribution, all the moments are infinite, so in particular the mean and variance are infinite. □

If a gamma Λ is used in conjunction with a Weibull $a(x)$ (instead of an exponential, whic was used in the previous example), then the model has a **Burr** (instead of a Pareto) distribution.

2.2 Splices

Another way of creating distributions is by **splicing** them. This means using different probability distributions on different intervals in such a way that the total probability adds up to 1.

For example, suppose larger loss sizes appear "**Pareto**"-ish, but smaller loss sizes are pretty uniform. You would like to build a model with a level density function for losses below 100, and then a declining density function starting at 100 which looks like a Pareto. Think a little bit about how you could do this.

You are going to use a distribution with a constant density function below 100, and a distribution function which looks like a Pareto above 100. By "looking like a Pareto", we mean it'll be of the form

$$f(x) = \frac{b_2 \alpha \theta^\alpha}{(\theta + x)^{\alpha+1}}$$

where b_2 is a constant that will make things work out right. You may have decided on what α and θ should be. Let's say $\alpha = 2$ and $\theta = 500$. Then the spliced distribution you will use has the form

$$f(x) = \begin{cases} b_1 & x < 100 \\ \dfrac{b_2(2)500^2}{(500 + x)^3} & x > 100 \end{cases}$$

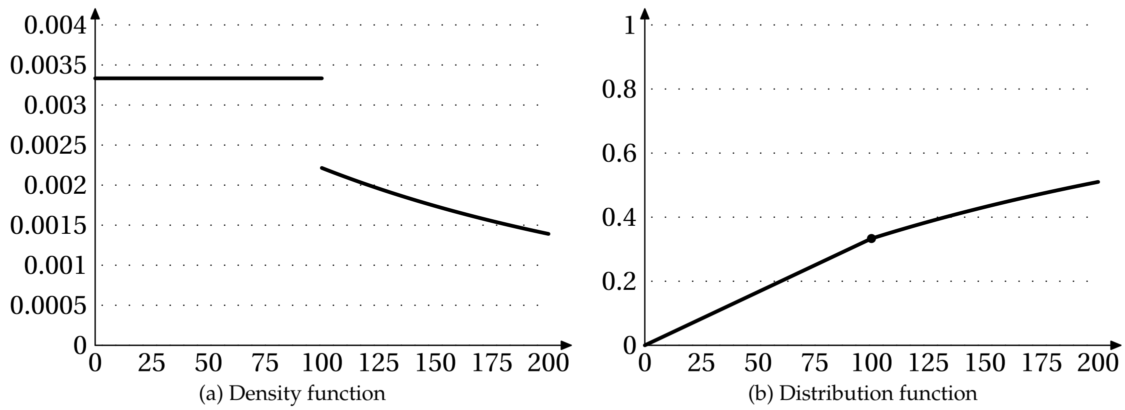


Figure 2.1: Spliced distribution with 1/3 weight below 100

How would you pick b_1 and b_2? One thing is absolutely necessary: the total probability, the probability of being less than 100 plus the probability of being greater than 100, must add up to 1.

For a spliced distribution, the sum of the probabilities of being in each component must add up to 1.

In our example, let's say that one third of all losses are below 100. Then you would want $F(100) = \frac{1}{3}$. You would set $b_1 = 1/300$ so that $\int_0^{100} b_1 \mathrm{d}x = \frac{1}{3}$. Without b_2, the Pareto distribution would have

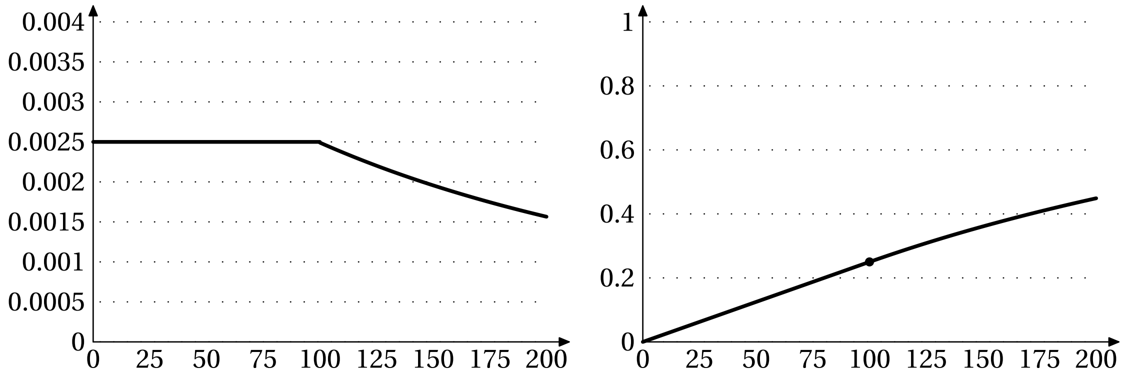
$$1 - F(100) = \left(\frac{500}{500+100}\right)^2 = \frac{25}{36}$$

However, you want $1 - F(100) = 1 - \frac{1}{3} = \frac{2}{3}$. Therefore, you would scale the Pareto distribution down by setting $b_2 = \frac{2/3}{25/36} = 0.96$. The spliced distribution has density function and distribution function

$$f(x) = \begin{cases} 1/300 & x < 100 \\ \dfrac{0.96(2)500^2}{(500+x)^3} & x > 100 \end{cases}$$
$$F(x) = \begin{cases} x/300 & x < 100 \\ 1 - 0.96\left(\dfrac{500}{500+x}\right)^2 & x > 100 \end{cases}$$

These are graphed in Figure 2.1. Notice that the density function is not continuous. Continuity of the density function is not a requirement of splicing. It may, however, be desirable. In the above example, suppose that we want the density function to be continuous. To allow this to happen, we will not specify the percentage of losses below 100, but will select it to make the density continuous. Then we would have to equate $f(100)$ for the uniform distribution, which is b_1, to $f(100)$ for the Pareto distribution, which is $2b_2(500^2)/600^3$. The other condition on b_1 and b_2 was developed above: since $100b_1$ is the probability of being below 100, we have

$$100b_1 = 1 - \left(\frac{25}{36}\right)b_2$$



(a) Density function (b) Distribution function

Figure 2.2: Continuous spliced distribution

Substituting the density equality for b_1 and solving, we get

$$\begin{aligned}
100\left(\frac{2b_2(500^2)}{600^3}\right) &= 1 - \left(\frac{25}{36}\right)b_2 \\
\left(\frac{200}{600}\right)\left(\frac{25}{36}\right)b_2 &= 1 - \left(\frac{25}{36}\right)b_2 \\
\left(\frac{25}{36}\right)\left(\frac{4}{3}\right)b_2 &= 1 \\
b_2 &= 1.08
\end{aligned}$$

It follows that $b_1 = 2(1.08)\left(\frac{500^2}{600^3}\right) = 1/400$ and $F(100) = \frac{1}{4}$. The density and distribution functions are

$$f(x) = \begin{cases} \dfrac{1}{400} & x < 100 \\ \dfrac{1.08(2)500^2}{(500+x)^3} & x > 100 \end{cases}$$

$$F(x) = \begin{cases} \dfrac{x}{400} & x < 100 \\ 1 - 1.08\left(\dfrac{500}{500+x}\right)^2 & x > 100 \end{cases}$$

These are graphed in Figure 2.2.

Quiz 2-1 The distribution of X is a spliced distribution. You are given:

(i) $\Pr(X \le 0) = 0$.
(ii) For $0 < x \le 100$, the distribution of X is uniform.
(iii) $F(100) = 1/3$.
(iv) For $x > 100$, the density function of X is

$$f(x) = \frac{\theta}{(\theta + x)^2}$$

Determine θ.

The textbook gives a formal definition of a spliced distribution as one whose density function is a weighted sum of density functions, with density function j having support (that is, nonzero) only on interval (c_{j-1}, c_j), with the intervals (c_{j-1}, c_j) disjoint, and weights a_j adding up to 1. Thus, in our first example with a uniform distribution below 100 and $\Pr(X < 100) = \frac{1}{3}$, and a Pareto distribution above 100, we would say that the splice distribution had two components:

1. A uniform distribution $f_1(x) = 0.01$ with weight $a_1 = \frac{1}{3}$ in the interval $(0, 100)$, and
2. A distribution defined by
$$f_2(x) = \frac{2(500^2)/(500+x)^3}{25/36} \qquad x > 100$$
with weight $a_2 = \frac{2}{3}$ in the interval $(100, \infty)$.

To use standard distributions with densities $g_j(x)$ (like Pareto distributions), divide their densities by $G_j(c_j) - G_j(c_{j-1})$, where $G_j(x)$ is the distribution function corresponding to $g_j(x)$.

Based on this definition of splice, every splice is a discrete mixture! It's a mixture of functions defined on disjoint intervals. If the functions are familiar, you may be able to use the formula sheet or your knowledge of the functions to evaluate moments.

Example 2F X follows a spliced distribution with the following density and distribution functions:

$$f(x) = \begin{cases} \dfrac{1}{400} & x < 100 \\ \dfrac{1.08(2)500^2}{(500+x)^3} & x > 100 \end{cases}$$
$$F(x) = \begin{cases} \dfrac{x}{400} & x < 100 \\ 1 - 1.08\left(\dfrac{500}{500+x}\right)^2 & x > 100 \end{cases}$$

Calculate the mean of X.

Solution: X can be considered a mixture of a uniform distribution on $[0, 100]$ with weight $1/4$ and a shifted Pareto distribution with weight $3/4$. The shifted Pareto distribution is shifted by 100, so set $y = x - 100$. Since $S(x) = 1.08\bigl(500/(500+x)\bigr)^2$ for $x > 100$, the conditional survival function is this survival function divided by $3/4$.

$$S(x \mid X > 100) = \frac{4}{3}(1.08)\left(\frac{500}{500+x}\right)^2 = 1.44\left(\frac{5}{6}\right)^2\left(\frac{600}{600+(x-100)}\right)^2 = \left(\frac{600}{600+y}\right)^2$$

which is a Pareto with $\theta = 600$, $\alpha = 2$. The mean of the shifted Pareto is 100 plus the mean of the unshifted Pareto, so

$$\mathbf{E}[X] = 0.25(50) + 0.75\left(100 + \frac{600}{2-1}\right) = \boxed{\mathbf{537.5}}$$

Exercises

Mixtures

2.1. You are given the following information about a portfolio of insurance risks:

- There are three classes of risks: A, B, and C.
- The number of risks in each class, and the mean and standard deviation of claim frequency for each class, are given in the following chart:

	Number	Claim Frequency	
	of		Standard
Class	Risks	Mean	Deviation
A	500	0.10	0.20
B	300	0.12	0.25
C	200	0.15	0.35

Determine the standard deviation of claim frequency for a risk randomly selected from the portfolio.

(A) Less than 0.240
(B) At least 0.240, but less than 0.244
(C) At least 0.244, but less than 0.248
(D) At least 0.248, but less than 0.252
(E) At least 0.252

2.2. A group of policyholders has three classes. The probability that policyholder is in each class, add mean and standard deviations for each class, are given in the following table.

Probability of class	Mean loss	Standard deviation of loss
0.5	10	12
0.3	20	30
0.2	30	60

The number of claims submitted by each policyholder is identically distributed for all classes.

1000 claims are submitted from this group. For each claim, the class of policyholder submitting the claim is unknown.

Using the normal approximation, calculate x such that there is a 95% probability that the sum of the claims is less than x.

Exercises continue on the next page . . .

2.3. For a group of 1000 policyholders in three classes, you are given:

Number of policyholders	Mean loss	Standard deviation of loss
500	10	12
300	20	30
200	30	60

The number of claims submitted by each policyholder is identically distributed for all policyholders.

1000 claims are submitted from this group.

Using the normal approximation, calculate x such that there is a 95% probability that the sum of the claims is less than x.

Each policyholder submits one claim.

Using the normal approximation, calculate x such that there is a 95% probability that the sum of the claims is less than x.

2.4. You are given a portfolio of 100 risks in two classes, A and B, each having 50 risks. The losses of the risks in class A have a mean of 10 and a standard deviation of 5. For the entire portfolio, the mean loss is 20 and the standard deviation is 15.

Calculate the standard deviation of losses for risks in class B.

(A) Less than 9
(B) At least 9, but less than 13
(C) At least 13, but less than 17
(D) At least 17, but less than 21
(E) At least 21

2.5. Losses for an insurance coverage follow a distribution which is a mixture of an exponential distribution with mean 10 with 75% weight and an exponential distribution with mean 100 with 25% weight.

Calculate the probability that a loss is greater than 50.

2.6. Losses for an insurance coverage follow a distribution which is a mixture of an exponential distribution with mean 5 and an exponential distribution with mean θ. The mean loss size is 7.5. The variance of loss size is 75.

Determine the coefficient of skewness of the loss distribution.

2.7. **[151-82-93:11]** (2 points) A population is equally divided into two classes of drivers. The number of accidents per individual driver is Poisson for all drivers.

For a driver selected at random from Class 1, the expected number of accidents is uniformly distributed over $(0.2, 1.0)$.

For a driver selected at random from Class 2, the expected number of accidents is uniformly distributed over $(0.4, 2.0)$.

For a driver selected at random from this population, determine the probability of zero accidents.

(A) 0.41 (B) 0.42 (C) 0.43 (D) 0.44 (E) 0.45

Exercises continue on the next page . . .

2.8. [**M-S05:34**] The distribution of a loss, X, is a two-point mixture:

(i) With probability 0.8, X has a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 100$.
(ii) With probability 0.2, X has a two-parameter Pareto distribution with $\alpha = 4$ and $\theta = 3000$.

Calculate $\Pr(X \le 200)$.

(A) 0.76 (B) 0.79 (C) 0.82 (D) 0.85 (E) 0.88

2.9. [**M-F06:39**] The random variable N has a mixed distribution:

(i) With probability p, N has a binomial distribution with $q = 0.5$ and $m = 2$.
(ii) With probability $1 - p$, N has a binomial distribution with $q = 0.5$ and $m = 4$.

Which of the following is a correct expression for $\text{Prob}(N = 2)$?

(A) $0.125p^2$
(B) $0.375 + 0.125p$
(C) $0.375 + 0.125p^2$
(D) $0.375 - 0.125p^2$
(E) $0.375 - 0.125p$

2.10. [**CAS3-F06:20**] An insurance company sells hospitalization reimbursement insurance. You are given:

(i) Benefit payment for a standard hospital stay follows a lognormal distribution with $\mu = 7$ and $\sigma = 2$.
(ii) Benefit payment for a hospital stay due to an accident is twice as much as a standard benefit.
(iii) 25% of all hospitalizations are for accidental causes.

Calculate the probability that a benefit payment will exceed $15,000.

(A) Less than 0.12
(B) At least 0.12, but less than 0.14
(C) At least 0.14, but less than 0.16
(D) At least 0.16, but less than 0.18
(E) At least 0.18

2.11. [**CAS3-F06:19**] In 2006, annual claim frequency follows a negative binomial distribution with parameters β and r. β follows a uniform distribution on the interval (0,2) and $r = 4$.

Calculate the probability that there is at least 1 claim in 2006.

(A) Less than 0.85
(B) At least 0.85, but less than 0.88
(C) At least 0.88, but less than 0.91
(D) At least 0.91, but less than 0.94
(E) At least 0.94

Frailty models

2.12. For a random variable X, you are given:

(i) $h(x \mid \Lambda) = 2\Lambda x$
(ii) Λ has an exponential distribution with mean 0.5.

Calculate $\mathbf{E}\left[X^2 \mid \Lambda = 0.49\right]$.

Exercises continue on the next page . . .

2.13. Survival time X for a population follows a distribution having the following properties:

(i) $h(x \mid \Lambda) = \Lambda x^2$

(ii) Λ follows an exponential distribution with mean 0.05.

Calculate median survival time for the population.

2.14. Survival time X for a population of 100-year olds follows a Weibull distribution with the following hazard rate function:

$$h(x \mid \Lambda) = \tfrac{1}{3}\Lambda x^{-2/3}$$

Λ varies over the population. Λ has a gamma distribution with parameters $\alpha = 5$, $\theta = 0.5$.

Calculate the median future lifetime for the population.

2.15. The conditional hazard rate of a random variable X given Θ is

$$h(x \mid \Theta) = 0.1\Theta$$

The probability density function of Θ is

$$f(\theta) = 100^2\theta e^{-100\theta} \quad \theta > 0$$

Calculate the median of X.

(A) Less than 150
(B) At least 150, but less than 250
(C) At least 250, but less than 350
(D) At least 350, but less than 450
(E) At least 450

Splices

2.16. **[CAS3-F06:18]** A loss distribution is a two-component spliced model using a Weibull distribution with $\theta_1 = 1{,}500$ and $\tau = 1$ for losses up to \$4,000, and a Pareto distribution with $\theta_2 = 12{,}000$ and $\alpha = 2$ for losses \$4,000 and greater. The probability that losses are less than \$4,000 is 0.60.

Calculate the probability that losses are less than \$25,000.

(A) Less than 0.900
(B) At least 0.900, but less than 0.925
(C) At least 0.925, but less than 0.950
(D) At least 0.950, but less than 0.975
(E) At least 0.975

2.17. Loss sizes follow a spliced distribution. The probability density function of the spliced distribution below 500 is the same as the probability density function of an exponential distribution with parameter $\theta = 250$. The probability density function of the spliced distribution above 500 is a multiple, a, of the probability density function of a Weibull distribution with parameters $\tau = 2$, $\theta = 400$.

Determine a.

Exercises continue on the next page . . .

2.18. Loss sizes follow a spliced distribution. Losses below 200 are uniformly distributed over $(0, 200]$. The probability density function of the spliced distribution above 200 is a multiple of the probability density function of an exponential distribution with parameter $\theta = 400$. The probability density function is continuous at 200.

Calculate the probability that a loss will be below 200.

2.19. Loss sizes follow a spliced distribution. The probability density function of this distribution below 200 is a multiple a of the probability density function of an exponential distribution with $\theta = 300$. The probability density function above 200 is the same as for an exponential distribution with $\theta = 400$.

Let X be loss size.

Calculate $\Pr(X < 100)$.

2.20. Loss sizes follow a spliced distribution. The probability density function of the spliced distribution below 100 is the same as that of a lognormal distribution with parameters $\mu = 3$, $\sigma = 2$. The probability density function of the spliced distribution above 100 is a times the probability density function of a two-parameter Pareto distribution with parameters $\alpha = 2$, $\theta = 300$.

Calculate the probability that a loss will be greater than 200.

2.21. The random variable X has the following spliced distribution:

$$F(x) = \begin{cases} \dfrac{x}{160} & 0 \le x \le 100 \\ 1 - 0.375e^{-(x-100)/200} & x > 100 \end{cases}$$

Calculate $\mathrm{Var}(X)$.

2.22. **[M-F05:35]** An actuary for a medical device manufacturer initially models the failure time for a particular device with an exponential distribution with mean 4 years.

This distribution is replaced with a spliced model whose density function:

(i) is uniform over $[0, 3]$
(ii) is proportional to the initial modeled density function after 3 years
(iii) is continuous

Calculate the probability of failure in the first 3 years under the revised distribution.

(A) 0.43 (B) 0.45 (C) 0.47 (D) 0.49 (E) 0.51

2.23. Loss sizes follow a spliced distribution. In the range $(0, 100)$, the probability density function is of the form $f(x) = c_1 e^{-x/\theta_1}$. In the range $(100, \infty)$, the probability density function is of the form $f(x) = c_2\theta_2^2/(\theta_2 + x)^3$. The parameters c_1, c_2, θ_1, and θ_2 are chosen so that

$$F(50) = 0.5$$
$$F(100) = 0.7$$
$$F(200) = 0.9$$

Determine $F(150)$.

(A) 0.80 (B) 0.81 (C) 0.82 (D) 0.83 (E) 0.84

Sample questions 15(a)–(c)

Solutions

2.1. This is a mixture distribution. The first moment is

$$\mathbf{E}[X] = 0.5(0.10) + 0.3(0.12) + 0.2(0.15) = 0.116$$

The second moment is

$$\mathbf{E}[X^2] = 0.5(0.10^2 + 0.20^2) + 0.3(0.12^2 + 0.25^2) + 0.2(0.15^2 + 0.35^2) = 0.07707$$

The variance is $0.07707 - 0.116^2 = 0.063614$. The standard deviation is $\sqrt{0.063614} = \boxed{0.252218}$. **(E)**

2.2. Since there are 1000 random claims "from the group", this is a sum of 1000 mixture random variables. The random variable for a single claim is a mixture with mean

$$0.5(10) + 0.3(20) + 0.2(30) = 17$$

and second moment

$$0.5(244) + 0.3(1300) + 0.2(4500) = 1412$$

so the variance is $1412 - 17^2 = 1123$. The mean and variance are multiplied by 1000 for 1000 claims, and the normal approximation requires

$$x = 1000(17) + 1.645\sqrt{1000(1123)} = \boxed{18{,}743.23}$$

2.3. Now we have a sum, not a mixture. The mean of the 1000 claims is the same as before, although technically it's calculated as

$$\mathbf{E}[X] = 500(10) + 300(20) + 200(30) = 17{,}000$$

The variance is the sum of the variances, or

$$\mathrm{Var}(X) = 500(12^2) + 300(30^2) + 200(60^2) = 1{,}062{,}000$$

The normal approximation requires

$$x = 17{,}000 + 1.645\sqrt{1{,}062{,}000} = \boxed{18{,}695.23}$$

The variance is lower than in the previous exercise (where it is 1123 per claim, or 1,123,000 for 1000 claims) because the uncertainty on who is submitting the claims has been removed.

2.4. Let μ_B be the mean loss for class B, and let σ_B^2 be the variance of loss for class B. Since the classes are equal in size, the raw moments of loss for the entire portfolio are equally weighted raw moments of the losses for each class. The first moment of losses for the entire portfolio is $20 = \frac{1}{2}(10 + \mu_B)$, so $\mu_B = 30$. The second moment of losses for the entire portfolio is

$$20^2 + 15^2 = \frac{1}{2}\left(10^2 + 5^2 + 30^2 + \sigma_B^2\right)$$
$$625 = \frac{1}{2}\left(1025 + \sigma_B^2\right)$$

from which it follows that $\sigma_B = \boxed{15}$. **(C)**

2.5. The probability that a loss is greater than 50 is the weighted average of the probability that each component of the mixture is greater than 50. Let X be a loss. If X_1 is exponential with mean 10 and X_2 is exponential with mean 100, then

$$\Pr(X_1 > 50) = e^{-50/10} = 0.006738$$
$$\Pr(X_2 > 50) = e^{-50/100} = 0.606531$$
$$\Pr(X > 50) = 0.75(0.006738) + 0.25(0.606531) = \boxed{0.1567}$$

2.6. Let w be the weight of the exponential with mean θ. The second moment is $75 + 7.5^2 = 131.25$. From equating the first moments,

$$\begin{aligned} 5(1-w) + \theta w &= 7.5 \\ 5 - 5w + \theta w &= 7.5 \\ w(\theta - 5) &= 2.5 \qquad (*) \end{aligned}$$

From equating the second moments,

$$\begin{aligned} 50(1-w) + 2w\theta^2 &= 131.25 \\ -50w + 2w\theta^2 &= 81.25 \\ 2w(\theta^2 - 25) &= 81.25 \end{aligned}$$

Dividing the first moment equation into the second moment equation eliminates w:

$$\begin{aligned} 2(\theta+5) &= \frac{81.25}{2.5} \\ \theta &= \frac{81.25}{2.5(2)} - 5 = 11.25 \end{aligned}$$

Plugging into (*),

$$\begin{aligned} w(6.25) &= 2.5 \\ w &= \frac{2.5}{6.25} = 0.4 \end{aligned}$$

To calculate skewness, we only need $\mathbf{E}[X^3]$, since we already know the first and second moments and the variance. For an exponential, $\mathbf{E}[X^3] = 6\theta^3$, so

$$\begin{aligned} \mathbf{E}[X^3] &= 6\big(0.6(5^3) + 0.4(11.25^3)\big) = 3867.1875 \\ \gamma_1 &= \frac{\mathbf{E}[X^3] - 3\,\mathbf{E}[X^2]\mu + 2\mu^3}{\sigma^3} \\ &= \frac{3867.1875 - 3(131.25)(7.5) + 2(7.5^3)}{75^{3/2}} \\ &= \frac{3867.1875 - 2953.125 + 843.75}{75^{3/2}} = \boxed{\mathbf{2.70633}} \end{aligned}$$

2.7. The probability of 0 accidents is $e^{-\lambda}$. We integrate this over the uniform distribution for each class:

$$\begin{aligned} \text{Class 1:}\quad \frac{1}{0.8}\int_{0.2}^{1} e^{-\lambda}\,\mathrm{d}\lambda &= \frac{e^{-0.2} - e^{-1}}{0.8} = 0.5636 \\ \text{Class 2:}\quad \frac{1}{1.6}\int_{0.4}^{2} e^{-\lambda}\,\mathrm{d}\lambda &= \frac{e^{-0.4} - e^{-2}}{1.6} = 0.3344 \end{aligned}$$

The average is $\frac{1}{2}(0.5636 + 0.3344) = \boxed{\mathbf{0.4490}}$. **(E)**

2.8. The probability of the mixture is the mixed probability. For a two-parameter Pareto with $\alpha = 2$, $\theta = 100$, the probability is $\Pr(X \le 200 \mid \text{(i)}) = 1 - \left(\frac{100}{300}\right)^2 = \frac{8}{9}$. For a two-parameter Pareto with $\alpha = 4$, $\theta = 3000$, it is $\Pr(X \le 200 \mid \text{(ii)}) = 1 - \left(\frac{3000}{3200}\right)^4 = 0.227524$. The mixed probability is $\Pr(X \le 200) = 0.8(8/9) + 0.2(0.227524) = \boxed{\mathbf{0.756616}}$. **(A)**

2.9. $\Pr(N = 2)$ for the first component is $0.5^2 = 0.25$, and for the second component $\binom{4}{2}0.5^4 = 0.375$, so mixing them, $0.25p + 0.375(1-p) = \boxed{\mathbf{0.375 - 0.125}\boldsymbol{p}}$. **(E)**

2.10. This is a mixture distribution. If we let $F_1(x)$ be the standard hospital stay distribution and $F_2(x)$ the accident stay distribution, we add $\ln 2$ to scale the distribution up to $F_2(x)$, so $F_2(x)$ is lognormal with $\mu = 7 + \ln 2$ and $\sigma = 2$. Equivalently, we could calculate $S_2(7500)$. By the Law of Total Probability,

$$\begin{aligned}\Pr(X > 15{,}000) &= 0.75S_1(15{,}000) + 0.25S_2(15{,}000)\\ &= 0.75\left[1-\Phi\left(\frac{\ln 15{,}000 - 7}{2}\right)\right] + 0.25\left[1-\Phi\left(\frac{\ln 7{,}500 - 7}{2}\right)\right]\\ &= 0.75[1-\Phi(1.31)] + 0.25[1-\Phi(0.96)]\\ &= 0.75(1-0.9049) + 0.25(1-0.8315) = \boxed{\mathbf{0.1135}} \qquad \textbf{(A)}\end{aligned}$$

2.11. If N is the number of claims, $1 - \Pr(N=0) = 1 - \frac{1}{(1+\beta)^4}$. We integrate this over the mixing uniform distribution, which has density $\frac{1}{2}$.

$$\begin{aligned}1-\Pr(N=0) &= \frac{1}{2}\int_0^2 \left(1 - \frac{1}{(1+\beta)^4}\right) d\beta\\ &= \frac{1}{2}\left(2 + \left[\frac{1}{3(1+\beta)^3}\right]_0^2\right)\\ &= 1 + \frac{1}{2}\left(\frac{1}{81} - \frac{1}{3}\right) = \boxed{\mathbf{0.839506}} \qquad \textbf{(A)}\end{aligned}$$

2.12. Since $a(x) = 2x$ here, $A(x) = x^2$ and $S(x \mid \Lambda = 0.49) = e^{-0.49x^2} = e^{-(0.7x)^2}$, so this is a Weibull with $\theta = 1/0.7$ and $\tau = 2$. According to the formula sheet, the second moment is

$$\theta^2\Gamma(1 + 2/\tau) = \frac{1}{0.7^2} = \boxed{\mathbf{2.0408}}$$

2.13. $H(x \mid \Lambda) = \Lambda x^3/3$, so

$$\begin{aligned}S(x \mid \Lambda) &= e^{-\Lambda x^3/3}\\ S(x) &= \mathbf{E}_\Lambda[e^{-\Lambda x^3/3}]\\ &= M_\Lambda\left(\frac{-x^3}{3}\right)\\ &= \frac{1}{1 + 0.05(x^3/3)} = \frac{1}{1 + x^3/60}\end{aligned}$$

We calculate the median m from first principles:

$$\begin{aligned}S(m) &= 0.5\\ \frac{1}{1 + m^3/60} &= 0.5\\ \frac{m^3}{60} &= 1\\ m &= \sqrt[3]{60} = \boxed{\mathbf{3.9149}}\end{aligned}$$

2.14. This is a frailty model with $a(x) = \frac{1}{3}x^{-2/3}$, and

$$A(x) = \int_0^x \tfrac{1}{3}t^{-2/3}\mathrm{d}t = x^{1/3}$$

By equation (2.1), the marginal survival function is

$$\begin{aligned}
S(x) &= \mathbf{E}_\Lambda\Big[S(x \mid \Lambda)\Big] = \mathbf{E}_\Lambda\left[e^{-\Lambda x^{1/3}}\right] \\
&= M_\Lambda\left(-x^{1/3}\right) \\
&= \left(1 + \theta x^{1/3}\right)^{-\alpha} \\
&= \left(1 + 0.5x^{1/3}\right)^{-5} \\
&= \left(\frac{1}{1 + (0.125x)^{1/3}}\right)^5
\end{aligned}$$

Setting $S(x)$ equal to 0.5, the median is

$$\begin{aligned}
\left(\frac{1}{1 + (0.125x)^{1/3}}\right)^5 &= 0.5 \\
1 + (0.125x)^{1/3} &= \sqrt[5]{2} \\
(0.125x)^{1/3} &= \sqrt[5]{2} - 1 \\
0.125x &= \left(\sqrt[5]{2} - 1\right)^3 \\
x = 8\left(\sqrt[5]{2} - 1\right)^3 &= \boxed{\mathbf{0.026303}}
\end{aligned}$$

2.15. The conditional survival function is

$$S(x \mid \theta) = e^{-0.1\theta x}$$

and integrating over Θ,

$$S(x) = \mathbf{E}[e^{-0.1\Theta x}] = M_\Theta(-0.1x)$$

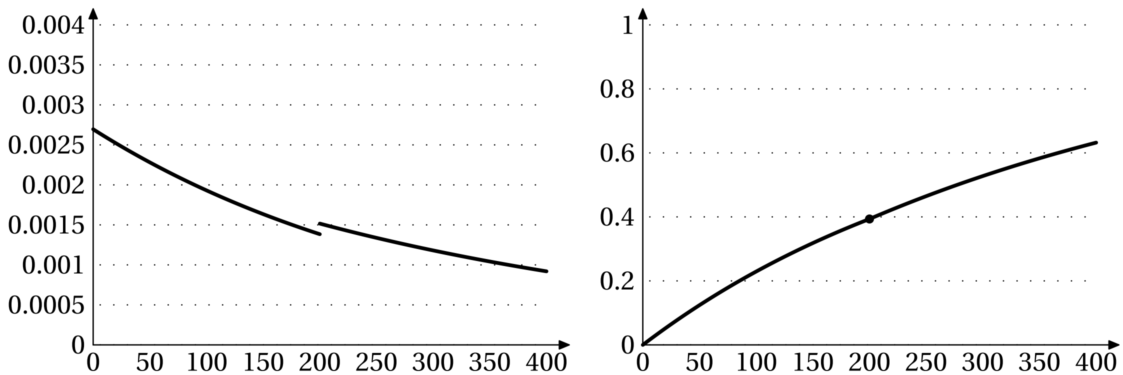
Since Θ's distribution is gamma with $\alpha = 2$ and $\theta = 0.01$,

$$S(x) = (1 + 0.001x)^{-2}$$

Setting this equal to 0.5,

$$\begin{aligned}
(1 + 0.001x)^2 &= 2 \\
0.001x &= \sqrt{2} - 1 \\
x = 1000(\sqrt{2} - 1) &= \boxed{\mathbf{414.2}} \qquad \textbf{(D)}
\end{aligned}$$

2.16. The Weibull is an exponential since $\tau = 1$, but we don't use it anyway. For an unspliced Pareto, $\Pr(X > 4000) = \left(\frac{12{,}000}{12{,}000+4{,}000}\right)^2 = 0.5625$, whereas here it is 0.4, so this Pareto's density is $0.4/0.5625$ times the normal density. Normally, $\Pr(X > 25{,}000) = \left(\frac{12{,}000}{37{,}000}\right)^2 = 0.105186$, so for the spliced Pareto, $\Pr(X > 25{,}000) = \frac{0.4}{0.5625}(0.105186) = 0.0748$, and $\Pr(X < 25{,}000) = 1 - 0.0748 = \boxed{\mathbf{0.9252}}$. **(C)**



(a) Density function (b) Distribution function

Figure 2.3: Spliced distribution in Exercise 2.19.

2.17. The probability of a loss below 500 is $1 - e^{-500/250} = 1 - e^{-2}$. Therefore, the probability of a loss above 500 is e^{-2}. Equating this to the Weibull at 500:

$$e^{-2} = ae^{-(500/400)^2}$$
$$a = e^{25/16-2} = e^{-7/16} = \boxed{\mathbf{0.645649}}$$

2.18. Let p be the probability, and a the multiple of the exponential distribution. Then $F(200) = p$, and that plus $\Pr(X > 200)$ must equal 1, so

$$p + ae^{-200/400} = 1$$

Since the density of the uniform distribution is $\frac{p}{200}$ and equals the exponential at 200,

$$\frac{p}{200} = \frac{ae^{-200/400}}{400}$$
$$a = 2pe^{1/2}$$

Substituting for a in the other expression:

$$p + 2p = 1$$
$$p = \boxed{\mathbf{\frac{1}{3}}}$$

2.19. Since $S(200) = e^{-200/400} = e^{-1/2}$, it follows that $a\left(1 - e^{-200/300}\right) = 1 - e^{-1/2}$. But then

$$a = \frac{1 - e^{-1/2}}{1 - e^{-2/3}} = \frac{0.393469}{0.486583} = 0.808638$$

and

$$\Pr(X < 100) = 0.808638\left(1 - e^{-100/300}\right) = \boxed{\mathbf{0.229224}}$$

The density and distribution functions are shown in Figure 2.3. Note that this is a discontinuous density function. Splicing two exponentials does not produce a single exponential.

2.20. We need to calculate a. First we calculate $F(100)$.

$$\Phi\left(\frac{\ln 100 - 3}{2}\right) = \Phi(0.80) = 0.7881$$

For the given Pareto, $S(100) = \left(\frac{3}{4}\right)^2 = 0.5625$. Therefore, a must be $(1 - 0.7881)/0.5625 = 0.3767$. Then

$$\Pr(X > 200) = 0.3767\left(\frac{300}{300 + 200}\right)^2 = (0.3767)(0.36) = \boxed{\mathbf{0.1356}}$$

2.21. X is a mixture with $5/8$ weight on a uniform on $[0, 100]$ and a $3/8$ weight on a shifted exponential. The uniform has mean 50 and variance $100^2/12$ and the exponential has $\theta = 200$ and shift 100, so its mean is 300 and its variance is 200^2. By the conditional variance formula and Bernoulli shortcut,

$$\begin{aligned}\operatorname{Var}(X) &= \mathbf{E}[100^2/12, 200^2] + \operatorname{Var}(50, 300)\\ &= \left(\frac{5}{8}\right)\left(\frac{100^2}{12}\right) + \left(\frac{3}{8}\right)(200^2) + \left(\frac{5}{8}\right)\left(\frac{3}{8}\right)(300 - 50)^2 = \boxed{\mathbf{30{,}169.27}}\end{aligned}$$

2.22. The density function past 3 is $\frac{c}{4}e^{-x/4}$ for some c. Let $f(x) = a$ be the density for $0 \le x \le 3$. To make $f(x)$ integrate to 1, we need

$$3a + ce^{-3/4} = 1$$

since $\int_3^\infty \frac{c}{4}e^{-x/4}\mathrm{d}x = ce^{-3/4}$. To make $f(x)$ continuous at 3, we need

$$a = \frac{c}{4}e^{-3/4}$$

Therefore, $ce^{-3/4} = 4a$. Plugging into the previous equation, $3a + 4a = 1$, so $a = 1/7$. The probability of failure in the first 3 years is $3a = 3/7 = \boxed{\mathbf{0.43}}$. **(A)**

2.23. $F(50)$ is extraneous, since we only need the distribution function from 100 on, which is a multiple of a Pareto; let k be the multiplier. To match $F(100)$ and $F(200)$, we need

$$0.3 = S(100) = k\left(\frac{\theta}{\theta + 100}\right)^2$$

$$0.1 = S(200) = k\left(\frac{\theta}{\theta + 200}\right)^2$$

Dividing the second equation into the first,

$$\begin{aligned}\left(\frac{\theta + 200}{\theta + 100}\right)^2 &= 3\\ \frac{\theta + 200}{\theta + 100} &= \sqrt{3}\\ \theta\sqrt{3} + 100\sqrt{3} &= \theta + 200\\ \theta &= \frac{200 - 100\sqrt{3}}{\sqrt{3} - 1} = 36.6025\\ \left(\frac{\theta}{\theta + 200}\right)^2 &= \left(\frac{36.6025}{236.6025}\right)^2 = 0.02393\\ k &= \frac{0.1}{0.02393} = 4.179\end{aligned}$$

So

$$F(150) = 1 - k\left(\frac{\theta}{\theta + 150}\right)^2 = 1 - 4.179\left(\frac{36.6025}{186.6025}\right)^2 = 1 - 0.16077 = \boxed{\mathbf{0.83923}} \qquad \textbf{(E)}$$

Quiz Solutions

2-1. Equate $1 - F(100)$ with the integral of $f(x)$ for $x > 100$.

$$\begin{aligned} 1 - F(100) &= \int_{100}^{\infty} \frac{\theta \, \mathrm{d}x}{(\theta + x)^2} \\ \frac{2}{3} &= -\frac{\theta}{\theta + x}\bigg|_{100}^{\infty} = \frac{\theta}{\theta + 100} \\ 3\theta &= 200 + 2\theta \\ \theta &= \boxed{200} \end{aligned}$$

Part VII

Practice Exams

It's time to take practice exams!

These three exams cover most of the important topics of the course. They have 6 questions apiece, including 1 workbook question apiece. They are weighted approximately in accordance with the syllabus, with some exceptions.

The questions are almost all original. One of the questions is similar to a released exam question, with numbers changed, but I couldn't think of any better way to ask on the topic.

Practice Exam 1

1. (*9 points*) An auto insurance bodily injury coverage is available with limits of 25,000, 100,000, and 500,000. Losses observed in 2022 were as follows:

	25,000 limit		100,000 limit		500,000 limit	
Range of losses	Number of losses	Total losses	Number of losses	Total losses	Number of losses	Total losses
1–25000	162	1,453,788	275	2,057,550	92	725,696
25000 limit	81	2,025,000				
25001-100000			88	3,614,600	55	2,149,510
100000 limit			12	1,200,000		
100001-500000					10	1,940,120
500000 limit					4	2,000,000

The basic limit is 25,000.

(a) (*2 points*) Calculate the increased limits factor for a policy limit of 500,000.

(b) (*2 points*) Risk loads are used for the calculation of ILF. The risk load is LAS/100,000.
Calculate the increased limits factor for a policy limit of 500,000.

(c) (*2 points*) A Pareto distribution is fitted to the data Using maximum likelihood estimation. The estimated parameters are:

$$\hat{\alpha} = 2 \qquad \hat{\theta} = 30{,}000$$

Calculate the increased limits factor for a policy limit of 500,000 based on this fit.

(d) (*2 points*) The data for the 500,000 limit only is fitted to a Pareto using maximum likelihood. The estimated parameters are:

$$\hat{\alpha} = 1.5 \qquad \hat{\theta} = 31{,}700$$

Perform a chi-squared test on the fit. Do not merge any of the four ranges.
Determine the p-value of the test.

(e) (1 point) Discuss two issues with combining data from different policy limits.

2. (11 points) Aggregate losses follow a compound Poisson distribution with annual Poisson parameter $\lambda = 2$. The loss size distribution is

x	$\Pr(X = x)$
0	0.30
1	0.35
2	0.20
3	0.15

(a) (2 points) Let $P_X(z)$ be the probability generating function for loss sizes.
Calculate $P_X(2)$.

(b) (1 point) Write an expression for $P_S(z)$, the probability generating function for aggregate losses. The expression should be of a form that can be evaluated on a calculator with addition, subtraction, multiplication, division, exponentiation, and logarithms only, and z should be the only required input.

(c) (3 points)

(i) Calculate $f_S(0)$.

(ii) $f_S(4) = 0.09815$, $f_S(5) = 0.06751$, $f_S(6) = 0.04453$.
Calculate $f_S(7)$.

(d) (3 points) There is a per-loss deductible of 1.5.

(i) Calculate the probability of exactly 2 nonzero payments in one year.

(ii) Calculate the variance of annual payments.

(e) (2 points) A reinsurance contract pays 80% of the excess of annual losses over 2.5
Calculate expected reinsurance payments.

3. (*10 points*) A sample of losses has 159 observations. The highest 14 observations are

13.9	14.1	14.4	14.9	15.7	16.0	17.0
22.6	25.7	30.0	35.2	44.6	50.6	59.9

The data is fit to a GPD using 13.9 as the threshold. The fitted values are $\xi = 0.3979358$, $\beta = 9.1780018$.

(a) (*3 points*)

(i) Calculate the empirical estimate of the 95% VaR.

(ii) Calculate the empirical estimate of the 95% Expected Shortfall.

(b) (*3 points*)

(i) Calculate the 95% VaR using the estimated GPD.

(ii) Calculate the 95% Expected Shortfall using the estimated GPD.

(c) (*3 points*)

(i) Using the same threshold, 13.9, calculate the Hill estimator of α.

(ii) Calculate the 95% VaR using the Hill estimator.

(d) (*1 point*) Discuss how the threshold is selected.

Exam questions continue on the next page . . .

4. (*10 points*) [Workbook question]

You are given the following sample observations:

12 15 30 50 200

These observations are fit to a Pareto distribution using maximum likelihood. The fitted values are

$$\hat{\alpha} = 3.8565 \qquad \hat{\theta} = 179.05$$

(a) (*4 points*)

(i) Express the information matrix in terms of α, θ, and the observations x_i.

(ii) Evaluate the information matrix using the values of $\hat{\alpha}$, $\hat{\theta}$, and the observations.

(iii) State the asymptotic covariance matrix.

(b) (*3 points*)

(i) Calculate the fitted mean of the Pareto distribution.

(ii) Construct a 95% normal confidence interval for the mean using the delta method.

(c) (*3 points*) The null hypothesis is

H_0: $\alpha = 1$, $\theta = 100$

(i) Using the likelihood ratio test, determine the significance level at which the null hypothesis is rejected.

(ii) Determine whether the null hypothesis or the maximum likelihood fit is preferred using AIC.

(iii) Determine whether the null hypothesis or the maximum likelihood fit is preferred using SBC.

5. (*12 points*) The size of claims on an insurance policy has a uniform distribution over the interval $[0, \beta + 1]$. The parameter β varies by insured and follows a Pareto distribution with $\alpha = 2$, $\theta = 1$.

An insured submits a claim of size 2.

(a) (*2 points*)

(i) Estimate β using maximum likelihood.

(ii) Using the estimate for β, calculate the expected value of the next claim.

For parts (b)–(d), Bayesian estimation methods are used.

(b) (*4 points*)

(i) The loss function is the square difference between actual and estimate.
Determine the Bayes prediction of the next claim.

(ii) The loss function is the absolute difference between actual and estimate.
Determine the Bayes prediction of the next claim.

(c) (*2 points*) Construct a 95% HPD credibility interval for β.

(d) (*3 points*) Calculate the variance of the next claim.

(e) (*1 point*) Explain why the Bühlmann credibility premium cannot be calculated.

Exam questions continue on the next page . . .

6. (*8 points*) An insurer has a portfolio of one-year policies. The claims triangle shows cumulative payments on claims through each development year. Assume that claims occur uniformly throughout the year and all claims are fully developed by the end of year 4.

Incremental Payments

Accident	Development Year				
Year	0	1	2	3	4
2019	750	150	50	25	25
2020	1000	100	100	20	
2021	1050	400	150		
2022	1400	200			
2023	1500				

Earned premiums for each year are

Year	Earned Premiums
2019	1500
2020	1800
2021	2000
2022	2500
2023	3000

The expected loss ratio for each year is 70%.

(a) (*2 points*) Calculate the outstanding claims reserve as of 12/31/2023 using the chain ladder method.

(b) (*3 points*) Calculate the discounted outstanding claims reserve as of 12/31/2023 using the chain ladder method and an interest rate of 0.04.

(c) (*3 points*) Calculate the discounted outstanding claims reserve as of 12/31/2023 using the Bornheutter-Ferguson method and an interest rate of 0.04.

Solutions to the above questions begin on page 833.

Appendices

Appendix A. Solutions to the Practice Exams

Practice Exam 1

1. **[Section 6.1]**

(a) For limit of 25,000, payments per policy are

$$\frac{1{,}453{,}788 + 2{,}057{,}550 + 725{,}696 + (81 + 88 + 12 + 55 + 10 + 4)(25{,}000)}{162 + 81 + 275 + 88 + 12 + 92 + 55 + 10 + 4} = \frac{10{,}487{,}034}{779} = 13{,}462.17$$

For limit of 500,000, payments per policy are

$$\frac{725{,}696 + 2{,}149{,}510 + 1{,}940{,}120 + 2{,}000{,}000}{92 + 55 + 10 + 4} = \frac{6{,}815{,}326}{161} = 42{,}331.22$$

The ILF is

$$\frac{42{,}331.22}{13{,}462.17} = \boxed{3.144456}$$

(b) The risk load is $13{,}462.17^2/100{,}000 = 1812.30$ for a limit of 25,000 and $42{,}331.22^2/100{,}000 = 17{,}919.32$ for a limit of 500,000. The ILF is

$$\frac{42{,}331.22 + 17{,}919.32}{13{,}462.17 + 1812.30} = \boxed{3.9445}$$

(c)

$$\mathbf{E}[X \wedge 25000] = \frac{30{,}000}{1}\left(1 - \frac{30{,}000}{55{,}000}\right) = 13{,}636.36$$

$$\mathbf{E}[X \wedge 500{,}000] = \frac{30{,}000}{1}\left(1 - \frac{30{,}000}{530{,}000}\right) = 28{,}301.89$$

$$\frac{28{,}301.89}{13{,}636.36} = \boxed{2.0755}$$

(d)

$$F(25{,}000) = 1 - \left(\frac{31{,}700}{56{,}700}\right)^2 = 0.581963$$

$$F(100{,}000) = 1 - \left(\frac{31{,}700}{131{,}700}\right)^2 = 0.881911$$

$$F(500{,}000) = 1 - \left(\frac{31{,}700}{531{,}700}\right)^2 = 0.985442$$

$92 + 55 + 10 + 4 = 161$. We have

Range of losses	O_i	E_i
1–25000	92	93.6961
25001–100000	55	48.2915
100001–500000	10	16.6686
Over 500000	4	2.3438