An Integrated Component of Your Study Manual Program

Actuarial University
Your Path to Success

Study Manual

GOAL
Practice. Quiz. Test. PASS

Virtual eFlashcards

Videos & More!

a/s/m

Exam FAM Study Manual

2nd Edition

Abraham Weishaus, Ph.D., FSA, CFA, MAAA

a/s/m
Actuarial Study Materials
Learning Made Easier

a/s/m

Exam FAM Study Manual

2nd Edition

Abraham Weishaus, Ph.D., FSA, CFA, MAAA

TO OUR READERS:

Please check A.S.M.'s web site at **www.studymanuals.com** for errata and updates. If you have any comments or reports of errata, please e-mail us at **mail@studymanuals.com**.

Welcome to Actuarial University

Actuarial University is a reimagined platform built around a more simplified way to study. It combines all the products you use to study into one interactive learning center.

You can find integrated topics using this network icon.

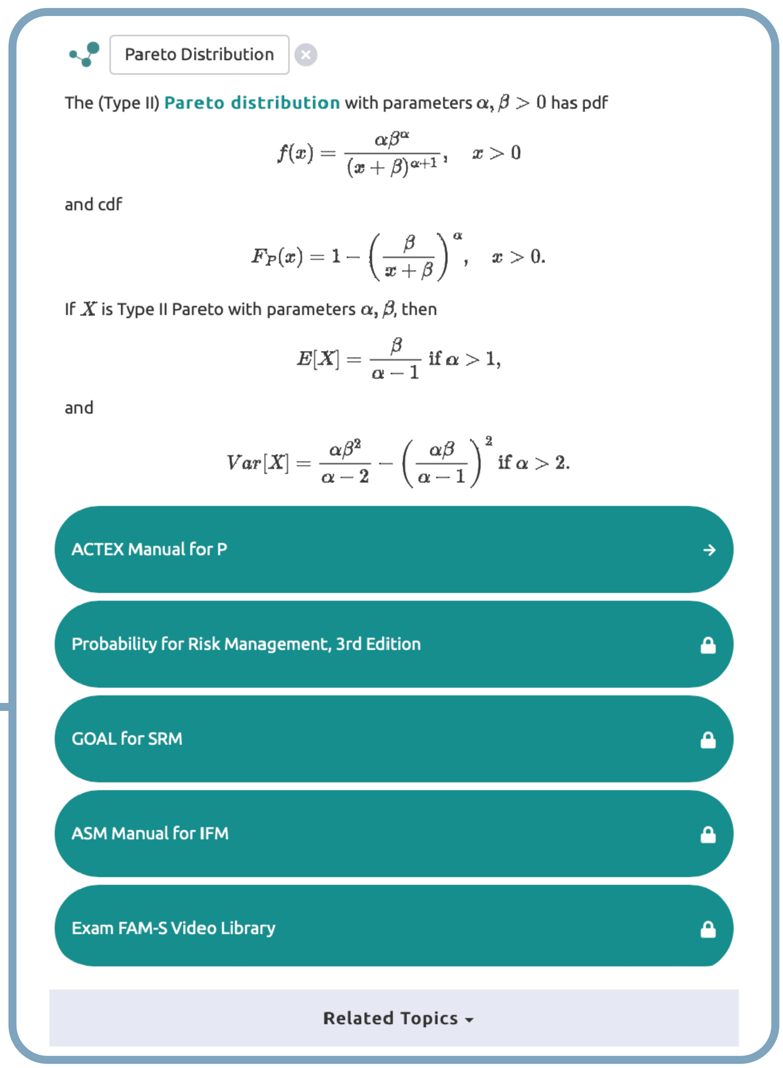
When this icon appears, it will be next to an important topic in the manual. Click the **link** in your digital manual, or search the underlined topic in your print manual.

1. Login to: www.actuarialuniversity.com

2. Locate the **Topic Search** on your exam dashboard and enter the word or phrase into the search field, selecting the best match.

3. A topic "**Hub**" will display a list of integrated products that offer more ways to study the material.

4. Here is an example of the topic **Pareto Distribution:**



Within the **Hub** there will be unlocked and locked products.

Unlocked Products are the products that you own.

ACTEX Manual for P →

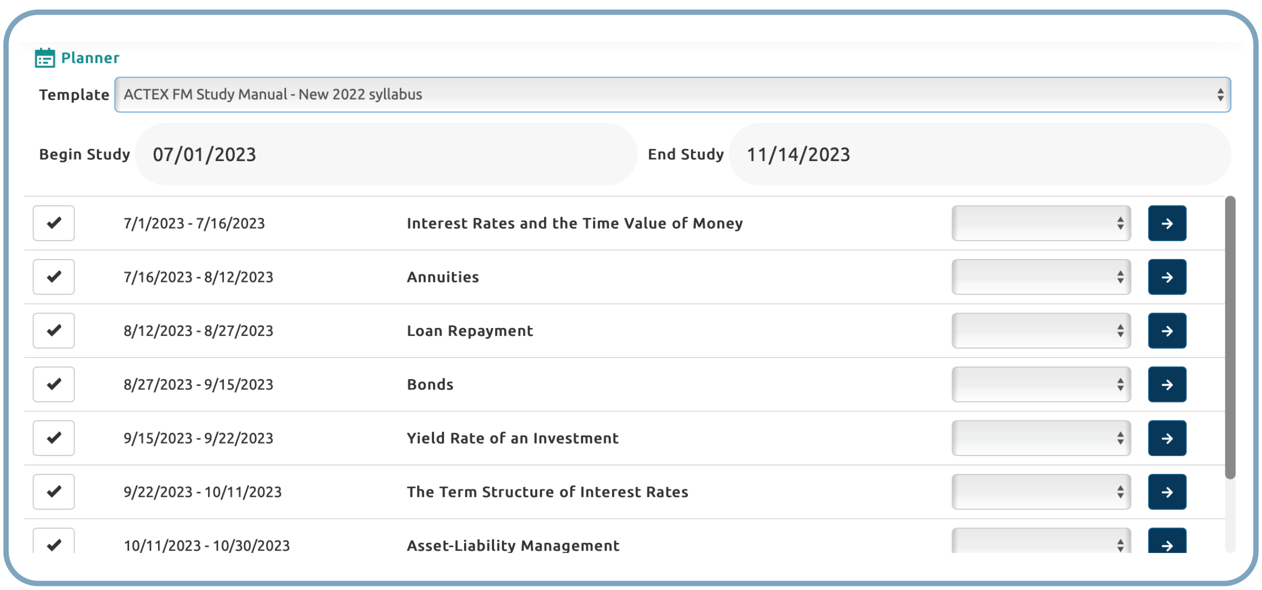
Locked Products are products that you do not own, and are available for purchase.

Probability for Risk Management, 3rd Edition

Many of Actuarial University's features are already unlocked with your study program, including:

Instructional Videos*	**Planner**
Topic Search	**Formula & Review Sheet**

Make your study session more efficient with our Planner!



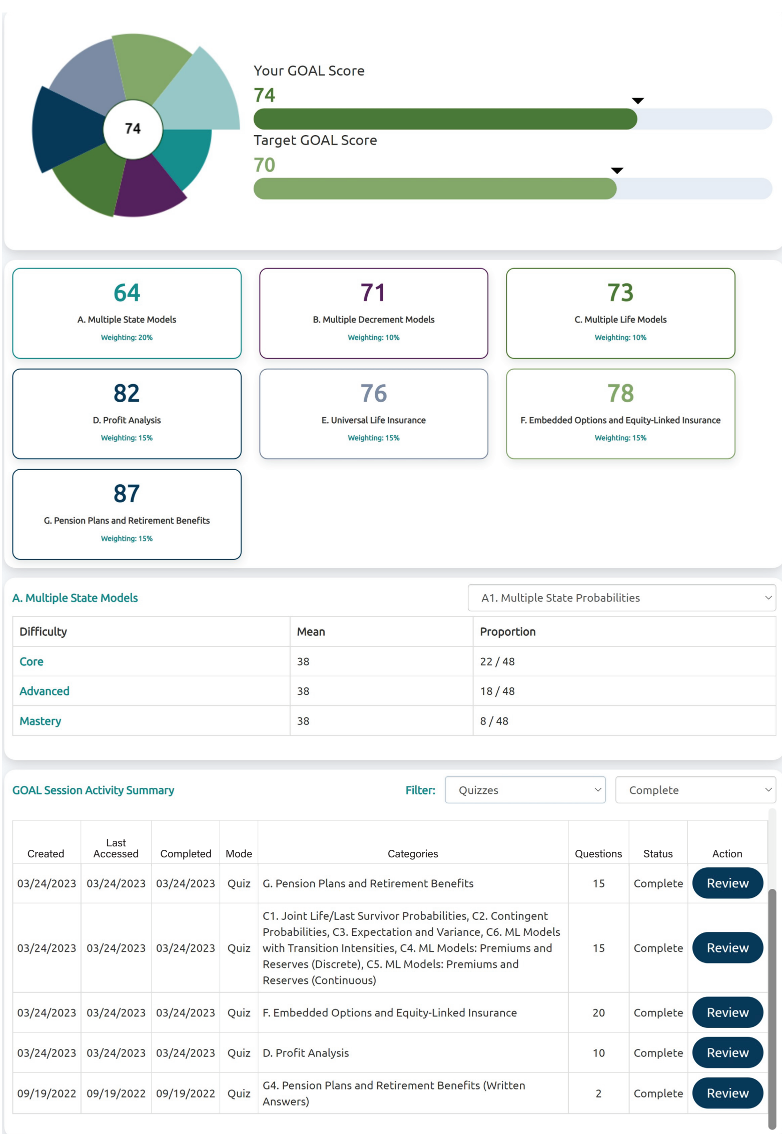
**Available standalone, or included with the Study Manual Program Video Bundle*

Track your exam readiness with GOAL Score!

Available for P, FM, FAM, FAM-L, FAM-S, ALTAM, ASTAM, MAS-I, MAS-II, & CAS 5

GOAL Score tracks your performance through GOAL Practice Sessions, Quizzes, and Exams, resulting in an aggregate weighted score that gauges your exam preparedness.

By measuring both your performance, and the consistency of your performance, **GOAL Score** produces a reliable number that will give you confidence in your preparation before you sit for your exam.



If your GOAL Score is a 70 or higher, you are well-prepared to sit for your exam!

See key areas where you can improve.

Detailed performance tracking.

Quickly return to previous sessions.

Contents

Preface

Welcome to Exam FAM!

Syllabus

The Fall 2024 syllabus is posted at the following URL:

https://www.soa.org/4a0bb9/globalassets/assets/files/edu/2024/fall/syllabi/2024-10-exam-fam-syllabus.pdf

The syllabus lists ten broad topics. The topics, their weights on the syllabus, and the lessons that cover the topics, are listed in the following table:

Topic	Weight	Lessons in Manual
Long Term Coverages and Retirement Financial Security Programs	2.5–5%	5
Mortality Models	10–15%	6–12
Present Value Random Variables for Long-Term Insurance Coverages	12.5–20%	13–24
Premium and Policy Value Calculation for Long-Term Insurance Coverages	15–22.5%	25–40
Short-Term Insurance and Reinsurance Coverages	5–10%	49–52, 54–55
Severity, Frequency, and Aggregate Models	12.5%–17.5%	2–4, 53, 56–62
Parametric Estimation	2.5%–7.5%	63–65
Introduction to Credibility	2.5%–5%	66–68
Pricing and Reserving for Short-Term Insurance Coverages	10–15%	46–48
Option Pricing Fundamentals	2.5–7.5%	42–44

The exam runs for 3.5 hours and has 34 multiple choice questions. Thus each 2.5% of syllabus weight represents 0.85 questions.

The material can be split into long term insurance and short term insurance, plus a section on option pricing.

The textbook for the long term insurance part and the option pricing section is *Actuarial Mathematics for Life Contingent Risks*. This is a well-written college style textbook with many exercises. The syllabus even says that exercises are part of the syllabus, something it doesn't say for the other syllabus books. However, many of the exercises require spreadsheets, so they are not realistic for a multiple-choice exam that does not provide a spreadsheet.

The main textbook for the short-term insurance part is *Loss Models* Fifth Edition, another college style textbook with many exercises. Some of the exercises are old exam questions reworded to the style of the book. Other exercises, however, may go well beyond a reasonable exam question.

Another textbook for the short-term insurance part is *Introduction to Ratemaking and Loss Reserving for Property and Casualty Insurance*. This textbook is more down-to-earth than the other two, and has some helpful exercises.

Other downloads from the SOA site

Tables

Download the tables you will be given on the exam. They will often be needed for the exercises. The tables for short-term insurances are currently posted at

https://www.soa.org/globalassets/assets/files/edu/2024/fall/2024-10-exam-fam-tables.pdf

The tables do not include a standard normal distribution table. Instead, the exam provides a Prometric normal distribution calculator, which computes the standard normal distribution function and its inverse to 5 decimal places. When working on the exercises, use Excel to calculate these functions. But in this manual, sometimes we don't maintain this precision.

The tables for long-term insurance are provided as an Excel workbook at

https://www.soa.org/4a1b9c/globalassets/assets/files/edu/2022/2022-10-exam-fam-l-tables-excel-workbook.xlsx

At the exam, you will be provided with digital files having the needed tables rather than with a workbook.

Notation and terminology note

The notation and terminology note is at

https://www.soa.org/4a0b3b/globalassets/assets/files/edu/2024/fall/study-note/2024-10-exam-fam-notation-note.pdf

You need not read this until you've finished studying all the material. Occasionally there are conflicts between the terminology or notation in different textbooks, or between the terminology used in the textbook and the terminology used at your job or in your school course, or between notation used in the U.S. versus notation used in the U.K. This notation and terminology note tells you which notation is used on the exam. (As you can guess, U.S. notation is used in preference to U.K. notation.) This manual uses the terminology that will be used on the exam.

The textbook uses the unusual name "policy value" for "reserve". On LTAM and earlier exams, the exams used the term "reserve", but the notation and terminology note states that "policy value" will be used on FAM for the expected value of future loss. "Reserve" will only be used for the capital a company puts aside to cover future losses.

Sample questions

Sample questions for short-term insurance and option pricing are at

https://www.soa.org/4a0b2c/globalassets/assets/files/edu/2024/fall/questions/2024-10-exam-fam-s-sample-questions.pdf

with solutions at

https://www.soa.org/4a10ac/globalassets/assets/files/edu/2024/fall/solutions/2024-10-exam-fam-s-sample-solutions.pdf

Sample questions for long-term insurance are at

https://www.soa.org/4a0b32/globalassets/assets/files/edu/2024/fall/questions/2024-10-exam-fam-l-sample-questions.pdf

with solutions at

https://www.soa.org/4a10a7/globalassets/assets/files/edu/2024/fall/solutions/2024-10-exam-fam-l-sample-solutions.pdf

I've handled short-term insurance and long-term insurance sample questions differently.

The sample questions for short-term insurance, from #1 to #77 (except for #64) are taken from old released exams given in the 2000–2006 period. (Yes, most of the syllabus hasn't changed for decades!) My manuals have always included any relevant released exam questions I could get my hands on as exercises, even old exam questions from before 2000. So I decided to simply add the new questions, #64 and #77–#102, to the exercises. These newer questions probably come from the question bank for this exam or its predecessors, questions they decided not to use any more, so treat them as real exam questions. The only question I have not included is #16, which is on a topic not on the syllabus and was mistakenly included in the list.[1]

[1] This is true at the time this was written, in August 2024. Perhaps question #16 will change to "DELETED" by the time you read this.

The sample questions for long-term insurance come from released MLC exams since 2012. They released these exams because they had written answer questions. I never included these in my manuals, preferring to provide an appendix with solutions to these exams. (There is still an appendix to this manual with solutions to LTAM exams.) I therefore provide at the end of each lesson a list of sample questions relevant to the lesson, rather than including these questions in exercise lists.

Old exam questions in this manual

There are about 610 original exercises in the manual and about 1080 questions from old exams or sample question lists. The old exam questions come from the following courses: SOA Part 4, CAS Part 4A, CAS Part 4B, SOA Course 150, SOA Course 151, 2000-syllabus Exam 3, Exam 4, Exam C, Exam M, and Exam MLC.

SOA Part 4 in 1986 had morning and afternoon sessions. The morning session had the more basic topics (through reserves), while the afternoon session had advanced topics (multiple lives, multiple decrements, etc.) Both sessions were multiple choice questions. Only morning session questions are relevant to Exam FAM.

SOA Course 150 from 1987 through 1991 had multiple choice questions in the morning and written answer questions in the afternoon.

Course 151 is the least relevant to this subject. I've only included a small number of questions from 151 in the first lesson, which is background.

CAS Parts 4A and 4B were administered in the 1990s. 4A covered long term insurance topics and 4B covered short-term insurance topics.

Back in 1999, the CAS and SOA created a sample exam for the then-new 2000 syllabus. This exam had some questions from previous exams but also some new questions, some of them not multiple choice. This sample exam was never a real exam, and some of its questions were defective. This sample exam is no longer available on the web. I have included appropriate questions from it. *Whenever an exercise is labeled 1999 C3 Sample or 1999 C4 Sample, it refers to the 1999 sample, not the current list of sample questions.*

Questions from old exams are marked xxx:yy, where xxx is the time the exam was given, with S for spring and F for fall followed by a 2-digit year, and yy is the question number. Sometimes xxx is preceded with SOA or CAS to indicate the sponsoring organization. From about 1986 to 2000, SOA exams had 3-digit numbers (like 150) and CAS exams were a number and a letter (like 4A). From 2000 to Spring 2003, the exams were jointly sponsored. There was a period in the 1990s when the SOA, while it allowed use of its old exam questions, did not want people to reveal which exam they came from. As a result, I sometimes had study notes for old exams in this period and could not identify the exam they came from. In such a case, I mark the question aaa-bb-cc:yy, where aaa-bb-cc is the study note number and yy is the question number. Generally aaa is the exam number (like 150), and cc is the 2-digit year the study note was published.

Characteristics of this exam

The exam will have 34 multiple choice questions. You will be given 3.5 hours to complete the exam, or 6.176 minutes per question.

There is no penalty for guessing. Fill in all questions regardless of whether you have time to work out the question or not—you lose nothing and you may be lucky!

The answer choices on SOA exams are almost always specific answers, not ranges, with the exception of some loss distribution questions.

Study schedule

Although this manual seems huge, much of it is exercises and practice exams. You do not have to do every exercise; do enough to gain confidence with the material. With intense studying, you should be able to cover all the material in 4 months.

It is up to you to set up a study schedule. Different students will have different speeds and different constraints, so it's hard to create a study schedule useful for everybody. However, I offer a sample 17-week (4 month) study

Table 1: 17 Week Study Schedule for Exam FAM

Subject	Lessons	Study Period	Hard/Long Lessons	Easy/Short Lessons	Skip if Short on Time
Probability Review	1–4	1 week			2.3, 2.4
Types of Long Term Products	5	0.5 weeks			
Survival Distributions	6–12	2 weeks	7, 11	6	10
Insurances	13–18	1 week	16	18	16
Annuities	19–24	1 week	19, 22	24	
Premiums	25–33	1.5 weeks	27, 32		
Policy Values, Part I	34–36	1 week			
Policy Values, Part II	37–40	1 week	38		
Option Pricing	42–44	1 week			43
Short Term Insurance Reserving and Ratemaking	46–48	1 week		48	
Coverage Modifications	49–55	2 weeks			53.4, 53.5, 55
Discrete Distributions	56–57	0.5 weeks		57	
Aggregate Loss Models	58–62	1.5 week	62	61	
Parametric Estimation	63–65	1 week			
Credibility	66–68	1 week			67

schedule, Table 1, as a guide. The amount of time you spend on this lesson depends on the strength of your probability background. You may decide to skip it and refer to it as needed.

The study schedule lists lessons that are either long or hard, as well as those that are short or easy or just background, so that you may better allocate your study time within the study periods provided for each subject. I've also listed lessons you may skip if short on time.

Acknowledgements

I would like to thank the SOA and CAS for allowing me to use their old exam questions. I'd also like to thank Harold Cherry for suggesting this manual and for providing three of the pre-2000 SOA exams and all of the pre-2000 CAS exams I used.

I would like to thank Bill Marella and the entire team at ArchiMedia Advantage for linking this manual to GOAL, providing you with additional material and software to help you master the material.

The creators of TeX, LaTeX, and its multitude of packages all deserve thanks for making possible the professional typesetting of this mathematical material.

Errata

Please report all errors you find in these notes to the author. You may send them to the publisher at mail@studymanuals.com or directly to me at errata@aceyourexams.net. Please identify the manual and edition the error is in. This is the 2nd edition of the Exam FAM manual.

An errata list will be posted at errata.aceyourexams.net. Check this errata list frequently.

Flashcards

Many students find flashcards a useful tool for learning key formulas and concepts. ASM flashcards, available from the same distributors that sell this manual, contain the formulas and concepts from this manual in a convenient deck of cards. The cards have cross references, usually by page, to the manual.

Lesson 2

Parametric Distributions

Reading: *Loss Models* Fifth Edition 4, 5.3—5.4

A ***parametric distribution*** is one that is defined by a fixed number of parameters. Examples of parametric distributions are the **exponential distribution** (parameter θ) and the **Pareto distribution** (parameters α, θ). Any distribution listed in the *Loss Models* appendix is parametric.

The alternative to a parametric distribution is a ***data-dependent distribution***. A data-dependent distribution is one where the specification requires at least as many "parameters" as the number of data points in the sample used to create it; the bigger the sample, the more "parameters".

It is common to use parametric distributions for short term insurance claim counts (**frequency**) and loss size (**severity**). Parametric distributions have many advantages. One of the advantages of parametric distributions which makes them so useful for severity is that they handle inflation easily.

2.1 Scaling

A parametric distribution is a member of a ***scale family*** if any positive multiple of the random variable has the same form. In other words, the distribution function of cX, for c a positive constant, is of the same form as the distribution function of X, but with different values for the parameters. Sometimes the distribution can be parametrized in such a way that only one parameter of cX has a value different from the parameters of X. If the distribution is parametrized in this fashion, so that the only parameter of cX having a different value from X is θ, and the value of θ for cX is c times the value of θ for X, then θ is called a ***scale parameter***.

All of the continuous distributions in the tables (Appendix A) are scale families. The parametrizations given in the tables are often different from those you would find in other sources, such as your probability textbook. They are parametrized so that θ is the scale parameter. Thus when you are given that a random variable has any distribution in the appendix and you are given the parameters, it is easy to determine the distribution of a multiple of the random variable.

The only distributions not parametrized with a scale parameter are the **lognormal** and the **inverse Gaussian**. Even though the inverse Gaussian has θ as a parameter, it is not a scale parameter. The parametrization for the lognormal given in the tables is the traditional one. *If you need to **scale a lognormal**, proceed as follows: if X is lognormal with parameters (μ, σ), then cX is lognormal with parameters $(\mu + \ln c, \sigma)$.*

To scale a random variable not in the tables, you'd reason as follows. Let $Y = cX$, $c > 0$. Then

$$F_Y(y) = \Pr(Y \le y) = \Pr(cX \le y) = \Pr\left(X \le \frac{y}{c}\right) = F_X\left(\frac{y}{c}\right)$$

One use of scaling is in handling inflation. In fact, handling inflation is the only topic in this lesson that is commonly tested directly. If loss sizes are inflated by $100r\%$, the **inflated loss variable** Y will be $(1 + r)X$, where X is the pre-inflation loss variable. For a scale family with a scale parameter, you just multiply θ by $(1 + r)$ to obtain the new distribution.

Example 2A Claim sizes expressed in dollars follow a two-parameter Pareto distribution with parameters $\alpha = 5$ and $\theta = 90$. A euro is worth \$1.50.

Calculate the probability that a claim will be for 20 euros or less. ■

Solution: If claim sizes in dollars are X, then claim sizes in euros are $Y = X/1.5$. The resulting euro-based random variable Y for claim size will be Pareto with $\alpha = 5$, $\theta = 90/1.5 = 60$. The probability that a claim will be no more than 20 euros is

$$\Pr(Y \le 20) = F_Y(20) = 1 - \left(\frac{60}{60+20}\right)^5 = \boxed{0.7627}$$

□

Example 2B Claim sizes in 2022 follow a lognormal distribution with parameters $\mu = 4.5$ and $\sigma = 2$. Claim sizes grow at 6% uniform inflation during 2023 and 2024.

Calculate $f(1000)$, the probability density function at 1000, of the claim size distribution in 2024. ■

Solution: If X is the claim size random variable in 2022, then $Y = 1.06^2 X$ is the revised variable in 2024. The revised lognormal distribution of Y has parameters $\mu = 4.5 + 2\ln 1.06$ and $\sigma = 2$. The probability density function at 1000 is

$$\begin{aligned} f_Y(1000) &= \frac{1}{\sigma(1000)\sqrt{2\pi}} e^{-(\ln 1000-\mu)^2/2\sigma^2} \\ &= \frac{1}{(2)(1000)\sqrt{2\pi}} e^{-[\ln 1000-(4.5+2\ln 1.06)]^2/2(2^2)} \\ &= (0.000199471)(0.518814) = \boxed{0.0001035} \end{aligned}$$

□

Example 2C Claim sizes expressed in dollars follow a lognormal distribution with parameters $\mu = 3$ and $\sigma = 2$. A euro is worth \$1.50.

Calculate the probability that a claim will be for 100 euros or less. ■

Solution: If claim sizes in dollars are X, then claim sizes in euros are $Y = X/1.5$. As discussed above, the distribution of claim sizes in euros is lognormal with parameters $\mu = 3 - \ln 1.5$ and $\sigma = 2$. Then

$$F_Y(100) = \Phi\left(\frac{\ln 100 - 3 + \ln 1.5}{2}\right) = \Phi(1.01) = \boxed{0.8438}$$

□

Example 2D Claim sizes X initially follow a distribution with distribution function:

$$F_X(x) = 1 - \frac{1}{e^{0.01x}(1+0.01x)} \qquad x > 0$$

Claim sizes are inflated by 50% uniformly.

Calculate the probability that a claim will be for 60 or less after inflation. ■

Solution: Let Y be the increased claim size. Then $Y = 1.5X$, so $\Pr(Y \le 60) = \Pr(X \le 60/1.5) = F_X(40)$.

$$F_X(40) = 1 - \frac{1}{1.4e^{0.4}} = \boxed{0.5212}$$

□

2.2 Common parametric distributions

The tables provide a lot of information about the distributions, but if you don't recognize the distribution, you won't know to use the table. Therefore, it is a good idea to be familiar with the common distributions.

You should familiarize yourself with the *form* of each distribution, but not necessarily the constants. The constant is forced so that the density function will integrate to 1. If you know which distribution you are dealing with, you can figure out the constant. To emphasize this point, in the following discussion, we will use the letter c for constants rather than spelling out what the constants are. You are not trying to recognize the constant; you are trying to recognize the form.

We will mention the means and variances or second moments of the distributions. You need not memorize any of these. The tables give you the **raw moments**. You can calculate the **variance** as $\mathbf{E}[X^2] - \mathbf{E}[X]^2$. However, for

The gamma function

The gamma function $\Gamma(x)$ is a generalization to real numbers of the factorial function, defined by

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \mathrm{d}u$$

For positive integers n,

$$\Gamma(n) = (n-1)!$$

The most important relationship for $\Gamma(x)$ that you should know is

$$\Gamma(x+1) = x\Gamma(x)$$

for any real number x.

EXAMPLE 2E Evaluate $\dfrac{\Gamma(8.5)}{\Gamma(6.5)}$.

SOLUTION:

$$\frac{\Gamma(8.5)}{\Gamma(6.5)} = \left(\frac{\Gamma(8.5)}{\Gamma(7.5)}\right)\left(\frac{\Gamma(7.5)}{\Gamma(6.5)}\right) = (7.5)(6.5) = \boxed{\mathbf{48.75}}$$

---

frequently used distributions, you may want to memorize the mean and variance to save yourself some time when working out questions.

We will graph the distributions. You are not responsible for graphs, but they may help you understand the distributions.

The tables occasionally use the **gamma function** $\Gamma(x)$ in the formulas for the moments. You should have a basic knowledge of the gamma function; if you are not familiar with this function, see the sidebar. The tables also define and use the **incomplete gamma** and **incomplete beta** functions, but you can get by without knowing them.[1]

2.2.1 Uniform

A **uniform distribution** has a constant density on $[d, u]$:

$$f(x; d, u) = \frac{1}{u-d} \qquad d \le x \le u$$

$$F(x; d, u) = \begin{cases} 0 & x \le d \\ \dfrac{x-d}{u-d} & d \le x \le u \\ 1 & x \ge u \end{cases}$$

You recognize a uniform distribution both by its finite **support**[2] and by the lack of an x in the density function.

Its moments are

$$\mathbf{E}[X] = \frac{d+u}{2}$$

$$\mathrm{Var}(X) = \frac{(u-d)^2}{12}$$

[1] But Excel provides these functions. if you are taking Exam ASTAM, an exam that provides you with a spreadsheet, you may need to use these functions.

[2] "Support" is the range of values for which the probability density function is nonzero.

Its mean, median, and midrange are equal. The best way to calculate the second moment is to add up the variance and the square of the mean. However, some students prefer to use the following easy-to-derive formula:

$$\mathbf{E}[X^2] = \frac{1}{u-d}\int_d^u x^2\,\mathrm{d}x = \frac{u^3-d^3}{3(u-d)} = \frac{u^2+ud+d^2}{3} \tag{2.1}$$

If $d=0$, then the formula reduces to $u^2/3$.

The uniform distribution is not directly in the tables, so I recommend you memorize the formulas for mean and variance. However, if $d=0$, then the uniform distribution is a special case of a beta distribution with $\theta=u$, $a=1$, $b=1$.

2.2.2 Beta

The probability density function of a **beta distribution** with $\theta=1$ has the form

$$f(x;a,b) = cx^{a-1}(1-x)^{b-1} \qquad 0 \le x \le 1$$

The parameters a and b must be positive. They may equal 1, in which case the corresponding factor is missing from the density function. Thus if $a=b=1$, the beta distribution is a uniform distribution.

You recognize a beta distribution both by its finite support—it's the only common distribution for which the density is nonzero only on a finite range of values—and by factors with x and $1-x$ raised to powers and no other use of x in the density function.

If θ is arbitrary, then the form of the probability density function is

$$f(x;a,b,\theta) = cx^{a-1}(\theta-x)^{b-1} \qquad 0 \le x \le \theta$$

The distribution function can be evaluated if a or b is an integer. The moments are

$$\mathbf{E}[X] = \frac{\theta a}{a+b}$$
$$\mathrm{Var}(X) = \frac{\theta^2 ab}{(a+b)^2(a+b+1)}$$

The mode is $\theta(a-1)/(a+b-2)$ when a and b are both greater than 1, but you are not responsible for this fact.

Figure 2.1 graphs four beta distributions with $\theta=1$ all having mean $2/3$. You can see how the distribution becomes more peaked and normal looking as a and b increase.

2.2.3 Exponential

The probability density function of an **exponential distribution** has the form

$$f(x;\theta) = ce^{-x/\theta} \qquad x \ge 0$$

θ must be positive.

You recognize an exponential distribution when the density function has e raised to a multiple of x, and no other use of x.

The distribution function is easily evaluated. The moments are:

$$\mathbf{E}[X] = \theta$$
$$\mathrm{Var}(X) = \theta^2$$

Figure 2.2 graphs three exponential distributions. The higher the parameter, the more weight placed on higher numbers.

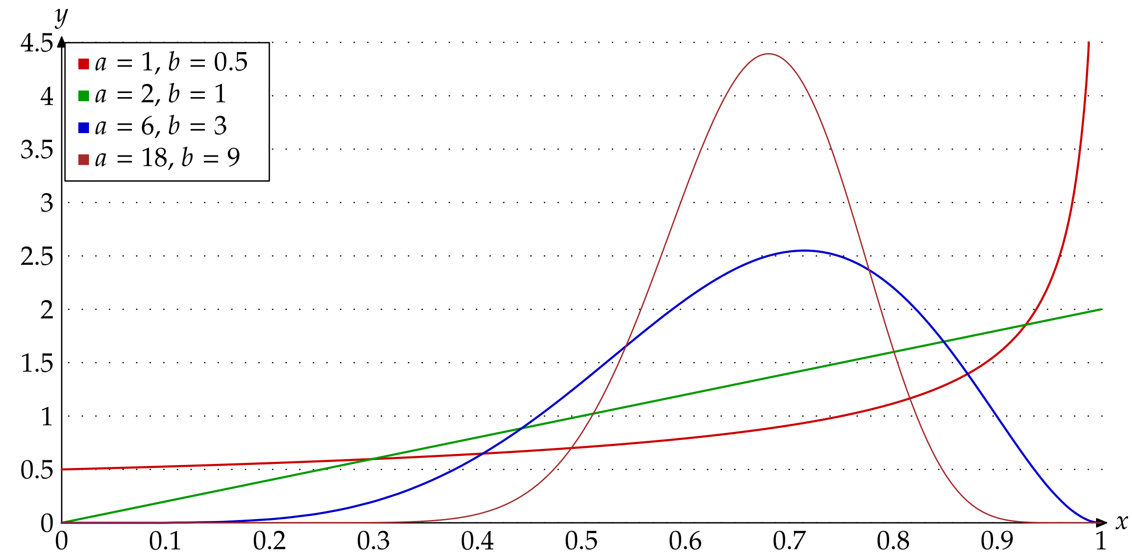


Figure 2.1: Probability density function of four beta distributions with $\theta = 1$ and mean 2/3

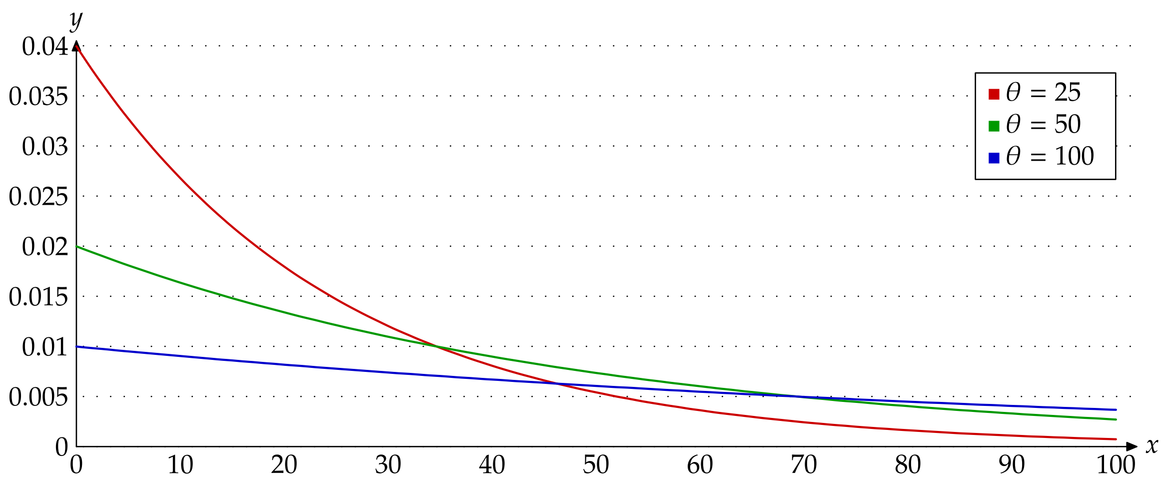


Figure 2.2: Probability density function of three exponential distributions

2.2.4 Weibull

A **Weibull distribution** is a transformed exponential distribution. If Y is exponential with mean μ, then $X = Y^{1/\tau}$ is Weibull with parameters $\theta = \mu^{1/\tau}$ and τ. An exponential is a special case of a Weibull with $\tau = 1$.

The form of the density function is

$$f(x;\tau,\theta) = cx^{\tau-1}e^{-(x/\theta)^\tau} \qquad x \ge 0$$

Both parameters must be positive.

You recognize a Weibull distribution when the density function has e raised to a multiple of a power of x, and in addition has a corresponding power of x, one lower than the power in the exponential, as a factor.

The distribution function is easily evaluated, but the moments require evaluating the **gamma function**, which usually requires numerical techniques. The moments are

$$\mathbf{E}[X] = \theta\Gamma(1 + 1/\tau)$$
$$\mathbf{E}[X^2] = \theta^2\Gamma(1 + 2/\tau)$$

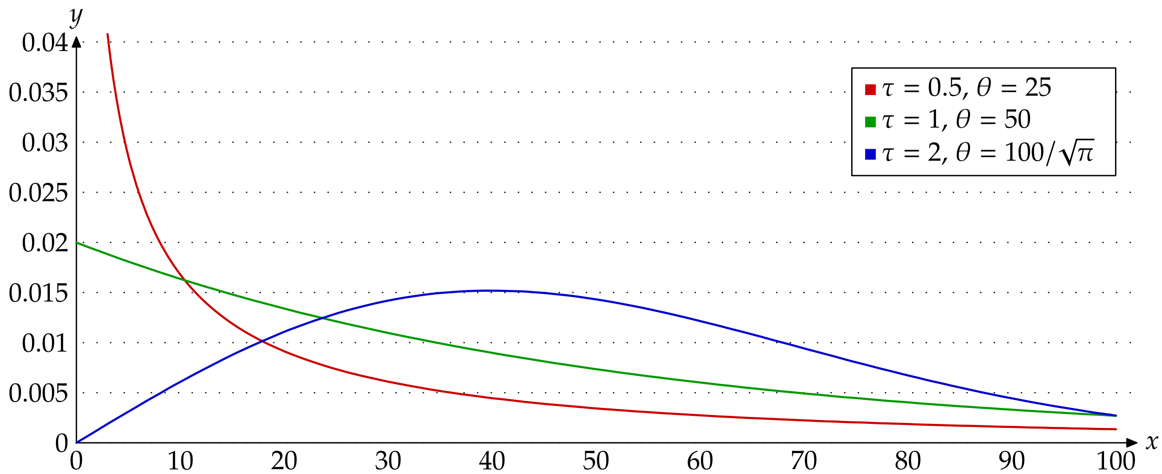


Figure 2.3: Probability density function of three Weibull distributions with mean 50

Figure 2.3 graphs three Weibull distributions with mean 50. The distribution has a non-zero mode when $\tau > 1$. Notice that the distribution with $\tau = 0.5$ puts a lot of weight on small numbers. To make up for this, it will also have to put higher weight than the other two distributions on very large numbers, so although it's not shown, its graph will cross the other two graphs for high x

2.2.5 Gamma

The form of the density function of a **gamma distribution** is

$$f(x; \alpha, \theta) = cx^{\alpha-1}e^{-x/\theta} \qquad x \geq 0$$

Both parameters must be positive.

When α is an integer, a gamma random variable with parameters α and θ is the sum of α independent exponential random variables with parameter θ. In particular, when $\alpha = 1$, the gamma random variable is exponential. The gamma distribution is called an **Erlang distribution** when α is an integer.

You recognize a gamma distribution when the density function has e raised to a multiple of x, and in addition has x raised to a power. Contrast this with a Weibull, where e is raised to a multiple of a *power* of x.

The distribution function may be evaluated if α is an integer; otherwise numerical techniques are needed. However, the moments are easily evaluated:

$$\mathbf{E}[X] = \alpha\theta$$
$$\mathrm{Var}(X) = \alpha\theta^2$$

Figure 2.4 graphs three gamma distributions with mean 50. As α goes to infinity, the graph's peak narrows and the distribution converges to a normal distribution.

The gamma distribution is one of the few for which the moment generating function has a closed form. In particular, the moment generating function of an exponential has a closed form. The only other distributions in the tables with closed form moment generating functions are the normal distribution (not actually in the tables, but the formula for the lognormal moments is the MGF of a normal) and the inverse Gaussian.

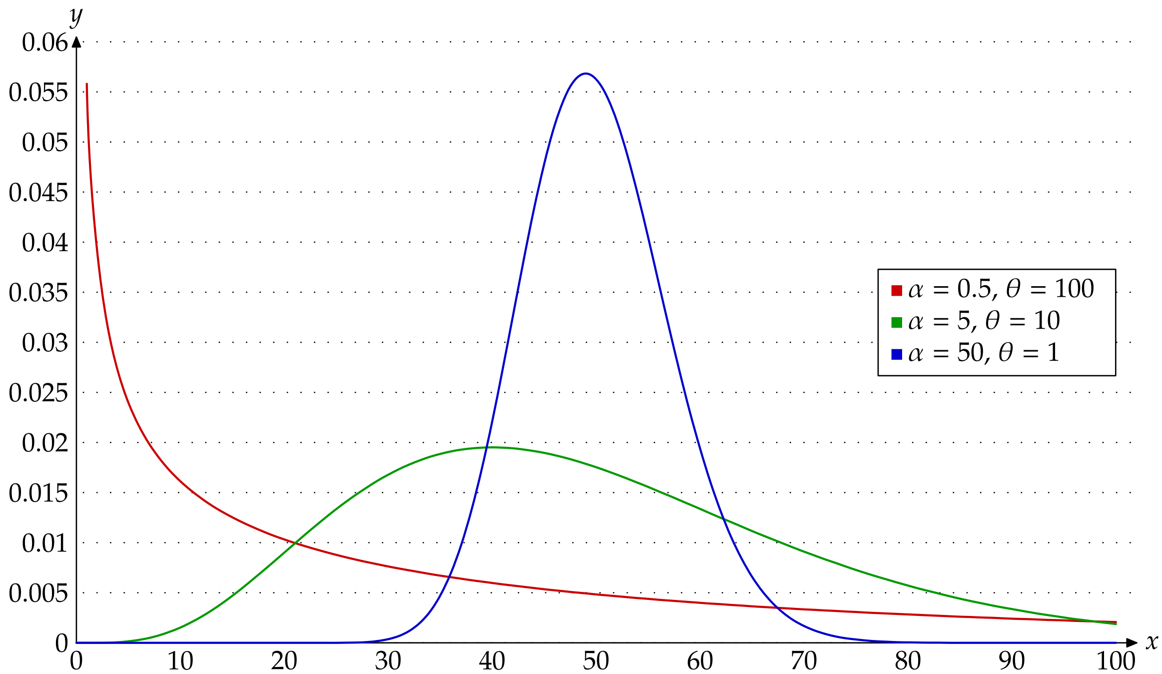


Figure 2.4: Probability density function of three gamma distributions with mean 50

2.2.6 Pareto

When we say "Pareto", we mean a *two-parameter* Pareto. On recent exams, they write out "two-parameter" to make it clear, but on older exams, you will often find the word "Pareto" with no qualifier. It always refers to a two-parameter Pareto, not a single-parameter Pareto.

The form of the density function of a **two-parameter Pareto** is

$$f(x) = \frac{c}{(\theta + x)^{\alpha+1}} \qquad x \geq 0$$

Both parameters must be positive.

You recognize a Pareto when the density function has a denominator with x plus a constant raised to a power.

The distribution function is easily evaluated. The moments are

$$\mathbf{E}[X] = \frac{\theta}{\alpha - 1} \qquad \alpha > 1$$
$$\mathbf{E}[X^2] = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} \qquad \alpha > 2$$

When α does not satisfy these conditions, the corresponding moments don't exist.

A shortcut formula for the variance of a Pareto is

$$\operatorname{Var}(X) = \mathbf{E}[X]^2\left(\frac{\alpha}{\alpha - 2}\right)$$

Figure 2.5 graphs three Pareto distributions, one with $\alpha < 1$ and the other two with mean 50. Although the one with $\alpha = 0.5$ puts higher weight on small numbers than the other two, its mean is infinite; it puts higher weight on large numbers than the other two, and its graph eventually crosses the other two as $x \to \infty$.

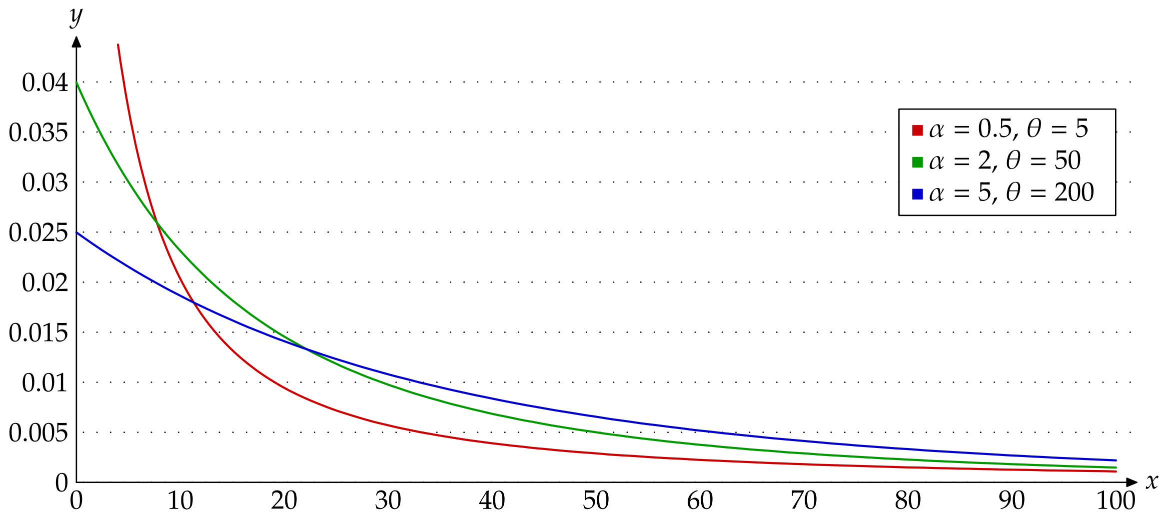


Figure 2.5: Probability density function of three Pareto distributions

2.2.7 Single-parameter Pareto

The form of the density function of a **single-parameter Pareto** is

$$f(x) = \frac{c}{x^{\alpha+1}} \qquad x \geq \theta$$

α must be positive. θ is not considered a parameter since it must be selected in advance, based on what you want the range to be.

You recognize a single-parameter Pareto by the range of nonzero values for its density function—unlike most other distributions, this range does not start at 0—and by the form of the density function, which has a denominator with x raised to a power. A beta distribution may also have x raised to a negative power, but its density function is 0 above a finite number.

A single-parameter Pareto X is a two-parameter Pareto Y shifted by θ: $X = Y + \theta$. Thus it has the same variance, and the mean is θ greater than the mean of a two-parameter Pareto with the same parameters.

$$\mathbf{E}[X] = \frac{\alpha\theta}{\alpha - 1} \qquad \alpha > 1$$
$$\mathbf{E}[X^2] = \frac{\alpha\theta^2}{\alpha - 2} \qquad \alpha > 2$$

2.2.8 Lognormal

The form of the density function of a **lognormal distribution** is

$$f(x) = \frac{ce^{-(\ln x - \mu)^2/2\sigma^2}}{x} \qquad x > 0$$

σ must be nonnegative.

You recognize a lognormal by the $\ln x$ in the exponent.

If Y is normal, then $X = e^Y$ is lognormal with the same parameters μ and σ. Thus, to calculate the distribution function, use

$$F_X(x) = F_Y(\ln x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

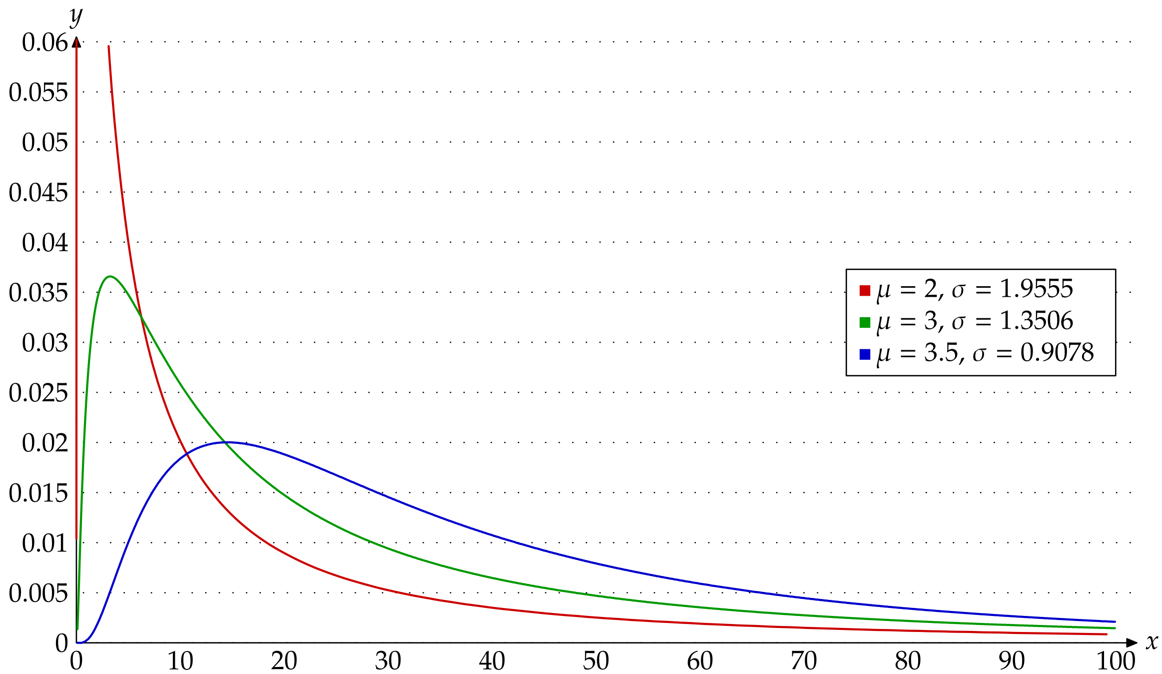


Figure 2.6: Probability density function of three lognormal distributions with mean 50

where $\Phi(x)$ is the standard normal distribution function, for which you are given tables. The moments of a lognormal are

$$\mathbf{E}[X] = e^{\mu+0.5\sigma^2}$$
$$\mathbf{E}[X^2] = e^{2\mu+2\sigma^2}$$

More generally, $\mathbf{E}[X^k] = \mathbf{E}[e^{kY}] = M_Y(k)$, where $M_Y(k)$ is the moment generating function of the corresponding normal distribution.

Figure 2.6 graphs three lognormals with mean 50. The mode is $\exp(\mu - \sigma^2)$, as stated in the tables. For $\mu = 2$, the mode is off the graph. As σ gets lower, the distribution flattens out.

Table 2.1 is a summary of the forms of probability density functions for common distributions.

2.3 The linear exponential family

The following material is based on *Loss Models* 5.4 which is on the syllabus. It won't play any role in this course, but the **linear exponential family** is commonly used for generalized linear models[3], which you'll study when working on Exam SRM. I doubt anything in this section will be tested on directly, so you may skip it.

A set of parametric distributions is in the linear exponential family if it can be parametrized with a parameter θ in such a way that in its density function, the only interaction between θ and x is in the exponent of e, which is x times a function of θ. In other words, its density function $f(x;\theta)$ can be expressed as

$$f(x;\theta) = \frac{p(x)e^{r(\theta)x}}{q(\theta)}$$

The set may have other parameters. $q(\theta)$ is the normalizing constant which makes the integral of f equal to 1. $r(\theta)$ is called the *canonical parameter* of the distribution.

[3]Many textbooks leave out "linear" and just call it the exponential family.

Table 2.1: Forms of probability density functions for common distributions

Distribution	Probability density function	
Uniform	c	$d \le x \le u$
Beta	$cx^{a-1}(\theta - x)^{b-1}$	$0 \le x \le \theta$
Exponential	$ce^{-x/\theta}$	$x \ge 0$
Weibull	$cx^{\tau-1}e^{-x^\tau/\theta^\tau}$	$x \ge 0$
Gamma	$cx^{\alpha-1}e^{-x/\theta}$	$x \ge 0$
Pareto	$\dfrac{c}{(x+\theta)^{\alpha+1}}$	$x \ge 0$
Single-parameter Pareto	$\dfrac{c}{x^{\alpha+1}}$	$x \ge \theta$
Lognormal	$\dfrac{ce^{-(\ln x-\mu)^2/2\sigma^2}}{x}$	$x > 0$

Examples of the linear exponential family are:

Gamma distribution The pdf is

$$f(x;\mu,\sigma) = \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha}$$

Let $r(\theta) = -1/\theta$, $p(x) = x^{\alpha-1}$, and $q(\theta) = \Gamma(\alpha)\theta^\alpha$.

Normal distribution The pdf is

$$f(x;\theta) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

Let $\theta = \mu$. The denominator of the pdf does not have x or θ so it can go into $q(\theta)$ or into $p(x)$. The exponent can be expanded into

$$-\frac{x^2}{2\sigma^2} + \frac{x\theta}{\sigma^2} - \frac{\theta^2}{2\sigma^2}$$

and only the second summand involves both x and θ, and x appears to the first power. Thus we can set $p(x) = e^{-x^2/2\sigma^2}$, $r(\theta) = \theta/\sigma^2$, and $q(\theta) = e^{\theta^2/2\sigma^2}\sigma\sqrt{2\pi}$.

Discrete distributions are in the linear exponential family if we can express the probability function in the linear exponential form.

Poisson distribution For a Poisson distribution, the probability function is

$$f(x;\lambda) = e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\frac{e^{x\ln\lambda}}{x!}$$

We can let $\theta = \lambda$, and then $p(x) = 1/x!$, $r(\theta) = \ln\theta$, and $q(\theta) = e^\theta$.

The textbook develops the following formulas for the mean and variance of a distribution from the linear exponential family:

$$\begin{aligned}\mathbf{E}[X] = \mu(\theta) &= \frac{q'(\theta)}{r'(\theta)q(\theta)} = \frac{\big(\ln q(\theta)\big)'}{r'(\theta)}\\ \mathrm{Var}(X) = v(\theta) &= \frac{\mu'(\theta)}{r'(\theta)}\end{aligned}$$

Thus, in the above examples:

Gamma distribution

$$\begin{aligned}\frac{\mathrm{d}\ln q}{\mathrm{d}\theta} &= \frac{\alpha}{\theta}\\ \frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{1}{\theta^2}\\ \mathbf{E}[X] &= \frac{\alpha/\theta}{1/\theta^2} = \alpha\theta\\ \mathrm{Var}(X) &= \frac{\alpha}{1/\theta^2} = \alpha\theta^2\end{aligned}$$

Normal distribution

$$\begin{aligned}\big(\ln q(\theta)\big)' &= \frac{2\theta}{2\sigma^2} = \frac{\theta}{\sigma^2}\\ r'(\theta) &= \frac{1}{\sigma^2}\\ \mathbf{E}[X] &= \frac{\theta/\sigma^2}{1/\sigma^2} = \theta\\ \mathrm{Var}(X) &= \frac{1}{1/\sigma^2} = \sigma^2\end{aligned}$$

Poisson distribution

$$\begin{aligned}\big(\ln q(\theta)\big)' &= 1\\ r'(\theta) &= \frac{1}{\theta}\\ \mathbf{E}[X] &= \frac{1}{1/\theta} = \theta\\ \mathrm{Var}(X) &= \frac{1}{1/\theta} = \theta\end{aligned}$$

2.4 Limiting distributions

The following material is based on *Loss Models* 5.3.3. I don't think it has ever appeared on the exam and doubt it ever will.

In some cases, as the parameters of a distribution go to infinity, the distribution converges to another distribution. To demonstrate this, we will usually have to use the identity

$$\lim_{\alpha\to\infty}\left(1+\frac{r}{\alpha}\right)^{\alpha} = e^r$$

Equivalently, if c is a constant (not dependent on α), then

$$\lim_{\alpha\to\infty}\left(1+\frac{r}{\alpha}\right)^{\alpha+c} = \lim_{\alpha\to\infty}\left(1+\frac{r}{\alpha}\right)^{\alpha}\left(1+\frac{r}{\alpha}\right)^{c} = e^r$$

As a simple example (not in the textbook) of a limiting distribution, consider a **gamma distribution** with a fixed mean μ, and let $\alpha \to \infty$. Then $\theta = \mu/\alpha$. The **moment generating function** is

$$M(t) = (1-\theta t)^{-\alpha} = \frac{1}{\left(1-\frac{\mu t}{\alpha}\right)^{\alpha}}$$

and as $\alpha \to \infty$, the denominator goes to $e^{-\mu t}$, so $M(t) \to e^{\mu t}$, which is the moment generating function of the constant μ. So as $\alpha \to \infty$, the limiting distribution of a gamma is a distribution equal to the mean with probability 1.

As another example, let's carry out textbook exercise 5.21, which asks you to demonstrate that the limiting distribution of a **Pareto** with θ/α constant as $\alpha \to \infty$ is an exponential. Let $k = \theta/\alpha$. The density function of a Pareto is

$$\begin{aligned} f(x;\alpha,\theta) &= \frac{\alpha\theta^{\alpha}}{(\theta+x)^{\alpha+1}} = \frac{\alpha(\alpha k)^{\alpha}}{(\alpha k+x)^{\alpha+1}} \\ &= \frac{k^{\alpha}}{(k+x/\alpha)^{\alpha+1}} = \frac{1}{k\big(1+(x/k)/\alpha\big)^{\alpha+1}} \end{aligned}$$

and the limit as $\alpha \to \infty$ is $(1/k)e^{-x/k}$. That is the density function of an exponential with mean k. Notice that as $\alpha \to \infty$, the mean of the Pareto converges to k.

Table 2.2: Summary of Parametric Distribution Concepts

- If X is a member of a scale family with scale parameter θ with value s, then cX is in the same family and has the same parameter values as X except that the scale parameter θ has value cs.
- All distributions in the tables are scale families with scale parameter θ except for lognormal and inverse Gaussian.
- If X is lognormal with parameters μ and σ, then cX is lognormal with parameters $\mu + \ln c$ and σ.
- If X is normal with parameters μ and σ^2, then e^X is lognormal with parameters μ and σ.
- See Table 2.1 to learn the forms of commonly occurring distributions. Useful facts are

Uniform on [d,u]	$\mathbf{E}[X] = \dfrac{d+u}{2}$
	$\operatorname{Var}(X) = \dfrac{(u-d)^2}{12}$
Uniform on [0,u]	$\mathbf{E}[X^2] = \dfrac{u^2}{3}$
Gamma	$\operatorname{Var}(X) = \alpha\theta^2$

- If Y is single-parameter Pareto with parameters α and θ, then $Y - \theta$ is two-parameter Pareto with the same parameters.
- X is in the linear exponential family if its probability density function can be expressed as

$$f(x;\theta) = \frac{p(x)e^{r(\theta)x}}{q(\theta)}$$

Exercises

2.1. Loss sizes for an insurance coverage follow an inverse gamma distribution with mean 6 and mode 4.

Calculate the coefficient of skewness for the losses.

(A) 3.1 (B) 3.2 (C) 3.3 (D) 3.4 (E) 3.5

2.2. For a commercial fire coverage

- In 2023, loss sizes follow a two-parameter Pareto distribution with parameters $\alpha = 4$ and θ.
- In 2024, there is uniform inflation at rate r.
- The 65^{th} percentile of loss size in 2024 equals the mean loss size in 2023.

Determine r.

2.3. **[CAS3-S06:26]** The aggregate losses of Eiffel Auto Insurance are denoted in euro currency and follow a lognormal distribution with $\mu = 8$ and $\sigma = 2$.

Given that 1 euro = 1.3 dollars, which set of lognormal parameters describes the distribution of Eiffel's losses in dollars?

(A) $\mu = 6.15$, $\sigma = 2.26$
(B) $\mu = 7.74$, $\sigma = 2.00$
(C) $\mu = 8.00$, $\sigma = 2.60$
(D) $\mu = 8.26$, $\sigma = 2.00$
(E) $\mu = 10.40$, $\sigma = 2.60$

2.4. **[4B-S90:37]** (2 points) Liability claim severity follows a Pareto distribution with a mean of 25,000 and parameter $\alpha = 3$.

If inflation increases all claims by 20%, the probability of a claim exceeding 100,000 increases by what amount?

(A) Less than 0.02
(B) At least 0.02, but less than 0.03
(C) At least 0.03, but less than 0.04
(D) At least 0.04, but less than 0.05
(E) At least 0.05

Exercises continue on the next page . . .

2.5. **[4B-F97:26]** (3 points) You are given the following:

- In 1996, losses follow a lognormal distribution with parameters μ and σ.
- In 1997, losses follow a lognormal distribution with parameters $\mu + \ln k$ and σ, where k is greater than 1.
- In 1996, $100p\%$ of the losses exceed the mean of the losses in 1997.

Determine σ.

Note: z_p is the $100p^{\text{th}}$ percentile of a normal distribution with mean 0 and variance 1.

(A) $2\ln k$

(B) $-z_p \pm \sqrt{z_p^2 - 2\ln k}$

(C) $z_p \pm \sqrt{z_p^2 - 2\ln k}$

(D) $\sqrt{-z_p \pm \sqrt{z_p^2 - 2\ln k}}$

(E) $\sqrt{z_p \pm \sqrt{z_p^2 - 2\ln k}}$

2.6. **[4B-S94:16]** (1 point) You are given the following:

- Losses in 1993 follow the density function

$$f(x) = 3x^{-4}, \quad x \geq 1,$$

  where x = losses in millions of dollars.
- Inflation of 10% impacts all claims uniformly from 1993 to 1994.

Determine the probability that losses in 1994 exceed 2.2 million.

(A) Less than 0.05

(B) At least 0.05, but less than 0.10

(C) At least 0.10, but less than 0.15

(D) At least 0.15, but less than 0.20

(E) At least 0.20

2.7. **[4B-F95:6]** (2 points) You are given the following:

- In 1994, losses follow a Pareto distribution with parameters $\theta = 500$ and $\alpha = 1.5$.
- Inflation of 5% impacts all losses uniformly from 1994 to 1995.

What is the median of the portion of the 1995 loss distribution above 200?

(A) Less than 600

(B) At least 600, but less than 620

(C) At least 620, but less than 640

(D) At least 640, but less than 660

(E) At least 660

Exercises continue on the next page . . .

2.8. **[CAS3-S04:34]** Claim severities are modeled using a continuous distribution and inflation impacts claims uniformly at an annual rate of i.

Which of the following are true statements regarding the distribution of claim severities after the effect of inflation?

1. An Exponential distribution will have scale parameter $(1+i)\theta$
2. A 2-parameter Pareto distribution will have scale parameters $(1+i)\alpha$ and $(1+i)\theta$.
3. A Paralogistic distribution will have scale parameter $\theta/(1+i)$

(A) 1 only (B) 3 only (C) 1 and 2 only (D) 2 and 3 only (E) 1, 2, and 3

2.9. **[CAS3-F05:21]** Losses during the current year follow a Pareto distribution with $\alpha = 2$ and $\theta = 400{,}000$. Annual inflation is 10%.

Calculate the ratio of the expected proportion of claims that will exceed \$750,000 next year to the proportion of claims that exceed \$750,000 this year.

(A) Less than 1.105
(B) At least 1.105, but less than 1.115
(C) At least 1.115, but less than 1.125
(D) At least 1.125, but less than 1.135
(E) At least 1.135

2.10. **[4B-S99:17]** You are given the following:

- In 1998, claim sizes follow a Pareto distribution with parameters θ (unknown) and $\alpha = 2$.
- Inflation of 6% affects all claims uniformly from 1998 to 1999.
- r is the ratio of the proportion of claims that exceed d in 1999 to the proportion of claims that exceed d in 1998.

Determine the limit of r as d goes to infinity.

(A) Less than 1.05
(B) At least 1.05, but less than 1.10
(C) At least 1.10, but less than 1.15
(D) At least 1.15, but less than 1.20
(E) At least 1.20

Exercises continue on the next page . . .

2.11. **[4B-F94:28]** (2 points) You are given the following:

- In 1993, the claim amounts for a certain line of business were normally distributed with mean $\mu = 1000$ and variance $\sigma^2 = 10{,}000$;

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \quad -\infty < x < \infty, \quad \mu = 1000, \ \sigma = 100.$$

- Inflation of 5% impacted all claims uniformly from 1993 to 1994.

What is the distribution for claim amounts in 1994?

(A) No longer a normal distribution
(B) Normal with $\mu = 1000$ and $\sigma = 102.5$.
(C) Normal with $\mu = 1000$ and $\sigma = 105.0$.
(D) Normal with $\mu = 1050$ and $\sigma = 102.5$.
(E) Normal with $\mu = 1050$ and $\sigma = 105.0$.

2.12. **[4B-S93:11]** (1 point) You are given the following:

i) The underlying distribution for 1992 losses is given by a lognormal distribution with parameters $\mu = 17.953$ and $\sigma = 1.6028$.
ii) Inflation of 10% impacts all claims uniformly the next year.

What is the underlying loss distribution after one year of inflation?

(A) Lognormal with $\mu' = 19.748$ and $\sigma' = 1.6028$.
(B) Lognormal with $\mu' = 18.048$ and $\sigma' = 1.6028$.
(C) Lognormal with $\mu' = 17.953$ and $\sigma' = 1.7631$.
(D) Lognormal with $\mu' = 17.953$ and $\sigma' = 1.4571$.
(E) No longer a lognormal distribution

2.13. X follows an exponential distribution with mean 10.
Determine the mean of X^4.

2.14. You are given

i) X is exponential with mean 2.
ii) $Y = X^{1.5}$.

Calculate $\mathbf{E}[Y^2]$.

2.15. X follows a gamma distribution with parameters $\alpha = 2.5$ and $\theta = 10$.
$Y = 1/X$.
Evaluate $\mathrm{Var}(Y)$.

Solutions

2.1. Looking up the tables for the inverse gamma distribution, we see that the mode is $\frac{\theta}{\alpha+1}$ and the mean is $\frac{\theta}{\alpha-1}$, so

$$\frac{\theta}{\alpha+1} = 4$$
$$\frac{\theta}{\alpha-1} = 6$$

Dividing the second line into the first,

$$\frac{\alpha-1}{\alpha+1}=\frac{4}{6}$$
$$\alpha=5 \qquad \theta=24$$

Then (γ_1 is the coefficient of skewness).

$$\begin{aligned}
\mathbf{E}\left[X^2\right] &= \frac{\theta^2}{(\alpha-1)(\alpha-2)} = \frac{24^2}{12} = 48\\
\operatorname{Var}(X) &= 48 - 6^2 = 12\\
\mathbf{E}\left[X^3\right] &= \frac{\theta^3}{(\alpha-1)(\alpha-2)(\alpha-3)} = \frac{24^3}{24} = 576\\
\gamma_1 &= \frac{576 - 3(48)(6) + 2(6^3)}{12^{1.5}}\\
&= \frac{144}{12^{1.5}} = \sqrt{12} = \boxed{3.4641} \qquad \textbf{(E)}
\end{aligned}$$

2.2. The mean in 2023 is $\theta/3$. By definition, the 65th percentile is the number π_{65} such that $F(\pi_{65}) = 0.65$, so $F(\theta/3) = 0.65$ for the 2024version of F. In 2024, F is two-parameter Pareto with inflated parameter $\theta' = (1+r)\theta$ and $\alpha = 4$, so

$$\begin{aligned}
1-\left(\frac{\theta'}{\theta'+(\theta/3)}\right)^4 &= 0.65\\
\frac{(1+r)\theta}{(1+r)\theta+\theta/3} &= \sqrt[4]{0.35}\\
\frac{1+r}{4/3+r} &= \sqrt[4]{0.35}\\
r(1-\sqrt[4]{0.35}) &= \frac{4}{3}\sqrt[4]{0.35}-1\\
r &= \frac{(4/3)\sqrt[4]{0.35}-1}{1-\sqrt[4]{0.35}} = \boxed{0.1107}
\end{aligned}$$

2.3. To scale a lognormal, you leave σ and add the logarithm of the scale to μ. So already, B and D are the only possibilities, and the question is whether you add or subtract $\ln 1.3$. But clearly the mean in dollars is higher than the mean in euros, so you add $\ln 1.3$ and get **(D)**.

2.4. Let X be the original variable, $Z = 1.2X$. Since the mean is 25,000, the parameter θ is $25{,}000(\alpha-1) = 50{,}000$.

$$\begin{aligned}
\Pr(X > 100{,}000) &= \left(\frac{50}{150}\right)^3 = \frac{1}{27}\\
\Pr(Z > 100{,}000) &= \left(\frac{60}{160}\right)^3 = \frac{27}{512}\\
\frac{27}{512}-\frac{1}{27} &= \boxed{0.0157} \qquad \textbf{(A)}
\end{aligned}$$

2.5. The key is to understand (iii). If (for example) 30% of losses exceed \$10000, what percentage does not exceed \$10000? (Answer: 70%) And what percentile of the distribution of losses is \$10000? (Answer: 70th). So statement (iii) is saying that the $100(1-p)^{\text{th}}$ percentile of losses in 1996 equals the mean of losses in 1997. Got it?

The mean of 1997 losses is $\exp(\mu + \ln k + \frac{\sigma^2}{2})$. The $100(1-p)$th percentile is $\exp(\mu - z_p\sigma)$. So:

$$\mu - z_p\sigma = \mu + \ln k + \frac{\sigma^2}{2}$$

$$\frac{\sigma^2}{2} + \sigma z_p + \ln k = 0$$

$$\sigma = \boxed{-z_p \pm \sqrt{z_p^2 - 2\ln k}} \qquad \textbf{(B)}$$

Notice that p must be less than 0.5, by the following reasoning. In general, the median of a lognormal (e^{μ}) is less than (or equal to, if $\sigma = 0$) the mean ($e^{\mu+\sigma^2/2}$), so the median of losses in 1996 is no more than the mean of losses in 1996, which in turn is less than the mean of losses in 1997 since $k > 1$, so $100p$ must be less than 50. Since p is less than 0.5, it follows that z_p will be negative, and σ is therefore positive, as it should be.

2.6. We recognize the 1993 distribution as a single-parameter Pareto with $\theta = 1$, $\alpha = 3$. The inflated parameters are $\theta = 1.1$, $\alpha = 3$. $\left(\frac{1.1}{2.2}\right)^3 = \boxed{0.125}$. **(C)**

2.7. Let X be the inflated variable, with $\theta = 525$, $\alpha = 1.5$. $\Pr(X > 200) = \left(\frac{525}{525+200}\right)^{1.5} = 0.6162$. Let F be the original distribution function, F^* the distribution of $X \mid X > 200$. Then $F(200) = 1 - 0.6162 = 0.3838$ and

$$F^*(x) = \Pr(X \le x \mid X > 200) = \frac{\Pr(200 < X \le x)}{\Pr(X > 200)} = \frac{F(x) - F(200)}{1 - F(200)}$$

So to calculate the median, we set $F^*(x) = 0.5$, which means

$$\frac{F(x) - F(200)}{1 - F(200)} = 0.5$$
$$\frac{F(x) - 0.3838}{0.6162} = 0.5$$
$$F(x) = 0.5(0.6162) + 0.3838 = 0.6919$$

We must find x such that $F(x) = 0.6919$.

$$1 - \left(\frac{525}{525 + x}\right)^{1.5} = 0.6919$$
$$\frac{525}{525 + x} = 0.4562$$
$$\frac{525 - 525(0.4562)}{0.4562} = x$$
$$x = \boxed{625.87} \qquad \textbf{(C)}$$

2.8. All the distributions are parameterized so that θ is the scale parameter and is multiplied by $1+i$; no other parameters change, and you should never divide by $1 + i$. Therefore **only 1** is correct. **(A)**

2.9. The Pareto is a scale distribution with scale parameter θ, so annual inflation of 10% increases θ by 10%, making it 440,000. The proportion of claims above 750,000 is $S(750{,}000) = \left(\frac{\theta}{\theta+750{,}000}\right)^{\alpha}$. Hence, the proportion this year is $\left(\frac{400{,}000}{400{,}000+750{,}000}\right)^2 = 0.120983$ and the proportion next year is $\left(\frac{440{,}000}{440{,}000+750{,}000}\right)^2 = 0.136714$. The ratio is $\frac{0.136714}{0.120983} = \boxed{1.1300}$. **(D)**

2.10. This is:

$$\frac{\left(\frac{1.06\theta}{1.06\theta+d}\right)^2}{\left(\frac{\theta}{\theta+d}\right)^2} = \frac{1.06^2(\theta+d)^2}{(1.06\theta+d)^2} \longrightarrow 1.06^2 = \boxed{1.1236}. \qquad \textbf{(C)}$$

2.11. If X is normal, then $aX + b$ is normal as well. In particular, $1.05X$ is normal. So the distribution of claims after 5% uniform inflation is normal.

For any distribution, multiplying the distribution by a constant multiplies the mean and standard deviation by that same constant. Thus in this case, the new mean is 1050 and the new standard deviation is 105. **(E)**

2.12. Add $\ln 1.1$ to μ: $17.953 + \ln 1.1 = 18.048$. σ does not change. **(B)**

2.13. The k^{th} moment for an exponential is given in the tables:

$$\mathbf{E}[X^k] = k!\theta^k$$

for $k = 4$ and the mean $\theta = 10$, this is $4!(10^4) = \boxed{\mathbf{240{,}000}}$.

2.14. While Y is Weibull, you don't need to know that. It's simpler to use $Y^2 = X^3$ and look up the third moment of an exponential.

$$\mathbf{E}[X^3] = 3!\theta^3 = 6(2^3) = \boxed{\mathbf{48}}$$

2.15. We calculate $\mathbf{E}[Y]$ and $\mathbf{E}[Y^2]$, or $\mathbf{E}[X^{-1}]$ and $\mathbf{E}[X^{-2}]$. Note that the special formula in the tables for integral moments of a gamma, $\mathbf{E}[X^k] = \theta^k(\alpha + k - 1)\cdots\alpha$ only applies when k is a *positive* integer, so it cannot be used for the -1 and -2 moments. Instead, we must use the general formula for moments given in the tables,

$$\mathbf{E}[X^k] = \frac{\theta^k\Gamma(\alpha + k)}{\Gamma(\alpha)}$$

For $k = -1$, this is

$$\mathbf{E}[X^{-1}] = \frac{\theta^{-1}\Gamma(\alpha - 1)}{\Gamma(\alpha)} = \frac{1}{\theta(\alpha - 1)}$$

since $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$. For $k = -2$,

$$\mathbf{E}[X^{-2}] = \frac{1}{\theta^2(\alpha - 1)(\alpha - 2)}$$

Therefore,

$$\operatorname{Var}(Y) = \frac{1}{10^2(1.5)(0.5)} - \left(\frac{1}{10(1.5)}\right)^2 = \boxed{\mathbf{0.00888889}}$$

Lesson 6

Survival Distributions: Probability Functions

Reading: *Actuarial Mathematics for Life Contingent Risks* 3rd edition 2.1, 2.2, 2.4, 3.1

We will study the probability distribution of future lifetime. Once we have specified the probability distribution, we will be able to answer questions like

> What is the probability that someone age 30 will survive to age 80?
>
> What is the probability that someone age 40 will die between ages 75 and 85?

With regard to the second question, in this course, whenever we say "between ages x and y", we mean between the x^{th} birthday and the y^{th} birthday. To say someone dies between ages 75 and 85 means that the person dies after reaching the 75^{th} birthday, but *before* reaching the 85^{th} birthday. If the person dies one month after his 85^{th} birthday, he has not died between ages 75 and 85.

For our survival models, we will use two styles of notation: probability (the type you use in probability courses, which writes arguments of functions with parentheses after the function symbol, like $f(x)$) and actuarial (which, as you will see, writes arguments as subscripts). We will use actuarial notation most of the time, but since you are probably already familiar with probability notation, we will start by discussing that, and then we'll define actuarial notation in terms of probability notation.

6.1 Probability notation

We first define T_x as the random variable for **time to death for someone age x**. Thus, for someone age 50, T_{50} is the amount of time until he dies, and to say $T_{50} = 32.4$ means that the person who was originally age 50 died when he was age 82.4, so that he lived exactly 32.4 years. We will use the symbol (x) to mean "someone age x", so (50) means "someone age 50". It is very common in this course to use the letter x to mean age.

In probability notation, $F_T(t)$ is the **cumulative distribution function** of T, or $\Pr(T \leq t)$. Usually, the cumulative distribution function is called the distribution function, dropping the word "cumulative". As an example of a cumulative distribution function, $F_{T_{50}}(30)$ is the probability that (50) does not survive 30 years. Rather than using a double subscript, we will abbreviate the notation for the cumulative distribution function of T_x as $F_x(t)$, or $F_{50}(30)$ in our example. The complement of the distribution function is called the ***survival function*** and is denoted by $S_T(t)$. In other words, $S_T(t) = \Pr(T > t)$. Thus $S_{T_{50}}(30)$ is the probability that (50) lives at least 30 years. In general, $S_T(t) = 1 - F_T(t)$. Once again, we'll abbreviate the notation as $S_x(t)$, or $S_{50}(30)$ in our example.

If we wanted to express the probability that (40) will die between ages 75 and 85 in terms of distribution functions, we would write it as

$$\Pr(35 < T_{40} \leq 45) = F_{40}(45) - F_{40}(35)$$

and if we wanted to express it in terms of survival functions, we'd write

$$\Pr(35 < T_{40} \leq 45) = S_{40}(35) - S_{40}(45)$$

In the probability expressions, you may wonder why the inequality is strict on one side and not on the other. The inequalities are strict or non-strict to be consistent with the definitions of F and S. However, we will always assume that T_x is a continuous random variable, so it doesn't matter whether the inequalities are strict or not.

Since we mentioned continuity as a property of our survival functions, let's discuss required and desirable characteristics of a survival function. **A survival function must have the following properties**:

1. $S_x(0) = 1$. Negative survival times are impossible.

2. $S_x(t) \geq S_x(u)$ for $u > t$. The function is monotonically nonincreasing. The probability of surviving a longer amount of time is never greater than the probability of surviving a shorter amount of time.
3. $\lim_{t\to\infty} S_x(t) = 0$. Eventually everyone dies; T_x is never infinite.

Those are the required properties of a survival function. Examples of valid survival functions (although they may not represent human mortality) are

1. $S_x(t) = e^{-0.01t}$
2. $S_x(t) = \dfrac{x+1}{x+1+t}$
3. $S_x(t) = \begin{cases} 1-0.01t & t \leq 100 \\ 0 & t > 100 \end{cases}$

Examples of invalid survival functions are

1. $S_x(t) = \begin{cases} 50-t & t \leq 50 \\ 0 & t > 50 \end{cases}$. Violates first property.
2. $S_x(t) = |\cos t|$. Violates second and third properties.

We will also assume the following properties for almost all of our survival functions:

1. $S_x(t)$ is differentiable for $t \geq 0$, with at most only a finite number of exceptions. Differentiability will allow us to define the probability density function (except at a finite number of points).
2. $\lim_{t\to\infty} tS_x(t) = 0$. This will assure that mean survival time exists.[1]
3. $\lim_{t\to\infty} t^2 S_x(t) = 0$. This will assure that the variance of survival time exists.

In the three examples of valid survival functions given above, you may verify that the first and third ones satisfy all of these properties but the second one does not satisfy the second and third properties.

Instead of saying "a person age x", we will often use the shorthand "(x)".

We would now like to relate the various T_x variables (one variable for each x) to each other. To do this, note that each T_x is a conditional random variable: it is the distribution of survival time, given that someone survived to age x. We can relate them using conditional probability:

$$\begin{aligned}&\text{Probability } (x) \text{ survives } t+u \text{ years}\\ &\qquad = \text{Probability } (x) \text{ survives } t \text{ years} \times \text{Probability } (x+t) \text{ survives } u \text{ years}\end{aligned}$$

or

$$\Pr(T_x > t+u) = \Pr(T_x > t)\Pr(T_{x+t} > u)$$
$$S_x(t+u) = S_x(t)S_{x+t}(u)$$
$$S_{x+t}(u) = \frac{S_x(t+u)}{S_x(t)} \tag{6.1}$$

In English: If you're given the survival function for (x), and you want to know the probability that someone t years older than x lives at least another u years, calculate the probability of (x) living at least $t+u$ years, and divide by the probability that (x) lives at least t years.

A special case is $x = 0$, for which (changing variables: the new x is the old t, the new t is the old u)

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} \tag{6.2}$$

[1] The expected value is $\int_0^\infty t f(t)\,dt$. Integrating by parts you get $-\lim_{t\to\infty} tS(t) + \int_0^\infty S(t)\,dt$.

The corresponding relationship for distribution functions is

$$\Pr(T_x \le t) = \frac{\Pr(T_0 \le x+t) - \Pr(T_0 \le x)}{\Pr(T_0 > x)}$$
$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{1 - F_0(x)} \tag{6.3}$$

Example 6A The survival function for newborns is

$$S_0(t) = \begin{cases} \sqrt{\frac{100-t}{100}} & t \le 100 \\ 0 & t > 100 \end{cases}$$

Calculate

1. The probability that a newborn survives to age 75 but does not survive to age 84.
2. The probability that (20) survives to age 75 but not to age 84.
3. $F_{60}(20)$.

■

Solution: Write each of the items we want to calculate in terms of survival functions.

1. We want $S_0(75) - S_0(84)$.

$$S_0(75) = \sqrt{\frac{25}{100}} = 0.5$$
$$S_0(84) = \sqrt{\frac{16}{100}} = 0.4$$
$$\Pr(75 < T_0 \le 84) = 0.5 - 0.4 = \boxed{\mathbf{0.1}}$$

2. We want $S_{20}(55) - S_{20}(64)$. We'll use equation (6.1) to calculate the needed survival functions.

$$S_{20}(55) = \frac{S_0(75)}{S_0(20)} = \frac{0.5}{\sqrt{80/100}} = 0.559017$$
$$S_{20}(64) = \frac{S_0(84)}{S_0(20)} = \frac{0.4}{\sqrt{80/100}} = 0.447214$$
$$\Pr(55 < T_{20} \le 64) = 0.559017 - 0.447214 = \boxed{\mathbf{0.111803}}$$

3. $F_{60}(20) = 1 - S_{60}(20)$, and

$$S_{60}(20) = \frac{S_0(80)}{S_0(60)} = \frac{\sqrt{0.2}}{\sqrt{0.4}} = \sqrt{0.5}$$
$$F_{60}(20) = 1 - \sqrt{0.5} = \boxed{\mathbf{0.292893}}$$

□

Quiz 6-1 (40) is subject to the survival function

$$S_{40}(t) = \begin{cases} 1 - 0.005t & t < 20 \\ 1.3 - 0.02t & 20 \le t \le 65 \end{cases}$$

Calculate the probability that (50) survives at least 30 years.

6.2 Actuarial notation

Actuarial notation puts arguments of functions in subscripts and sometimes superscripts before and after the base function symbol instead of using parenthesized arguments. We are going to learn two actuarial functions: p and q right now. Later on in this lesson, we'll also learn two other functions, d and l.

The first function we work with is $S_x(t) = \Pr(T_x > t)$. The actuarial symbol for this is ${}_tp_x$. The letter p denotes the concept of probability of survival. The x subscript is the age; the t presubscript is the duration.

The complement of the survival function is $F_x(t) = \Pr(T_x \le t)$. The actuarial symbol for this is ${}_tq_x$. The letter q denotes the concept of probability of death. A further refinement to this symbol is the probability of delayed death: the probability that T_x is between u and $u+t$, which is denoted by ${}_{u|t}q_x$.[2] For all three symbols, the t (but not the u) is usually omitted if it is 1.

To summarize the notation:

$$\begin{aligned} {}_tp_x &= S_x(t) \\ {}_tq_x &= F_x(t) \\ {}_{u|t}q_x &= F_x(t+u) - F_x(u) = S_x(u) - S_x(t+u) \end{aligned}$$

The **following relationships** are clear. For each one, an English translation is provided on the right. In these English translations, and indeed throughout this manual, the phrase "survives n years" means that the person does not die within n years. It does *not* mean that the person dies immediately at the end of n years.

$p_x = 1 - q_x$	The probability that a person age x survives one year is 1 minus the probability that the same person dies within one year.
${}_tp_x = 1 - {}_tq_x$	The probability that a person age x survives t years is 1 minus the probability that the same person dies within t years.
${}_tp_x\,{}_up_{x+t} = {}_{t+u}p_x$	The probability that a person age x survives t years and then survives for another u years is the probability that the same person survives $t+u$ years.
${}_{t\|u}q_x = {}_tp_x\,{}_uq_{x+t}$	The probability that a person age x dies at least t years from now but sooner than $t+u$ years from now is the probability that the same person survives t years and then dies within the next u years.

There are two additional useful formulas for ${}_{t|u}q_x$. The probability that (x) dies in the period from t to $t+u$ is the probability that (x) survives t years and does not survive $t+u$ years, or

$$ {}_{t|u}q_x = {}_tp_x - {}_{t+u}p_x \tag{6.4}$$

The probability that (x) dies in the period from t to $t+u$ is the probability that (x) dies within $t+u$ years minus the probability that (x) dies within t years, or

$$ {}_{t|u}q_x = {}_{t+u}q_x - {}_tq_x \tag{6.5}$$

EXAMPLE 6B You are given the following mortality table:

x	q_x
60	0.001
61	0.002
62	0.003
63	0.004
64	0.005

Calculate the probability that a person age 60 will die sometime between 2 and 5 years from now. ∎

[2] *Actuarial Mathematics for Life Contingent Risks* uses a large line on the baseline for this symbol, like ${}_{u}|_{t}q_x$. I think this is ugly, and older textbooks do not write it this way. Nor do recent exams.

Solution: The actuarial notation for what we are calculating is ${}_{2|3}q_{60}$. One way to calculate this is as the probability of living 2 years minus the probability of living 5 years, or ${}_2p_{60} - {}_5p_{60}$. We calculate:

$$\begin{aligned}
{}_2p_{60} &= p_{60}\,p_{61} \\
&= (1-q_{60})(1-q_{61}) = (1-0.001)(1-0.002) = 0.997002 \\
{}_5p_{60} &= {}_2p_{60}\,p_{62}\,p_{63}\,p_{64} \\
&= 0.997002(1-q_{62})(1-q_{63})(1-q_{64}) \\
&= 0.997002(1-0.003)(1-0.004)(1-0.005) = 0.985085 \\
{}_{2|3}q_{60} &= 0.997002 - 0.985085 = \boxed{0.011917}
\end{aligned}$$

In the following example, we'll relate actuarial and probability notation.

Example 6C You are given that

$$S_0(t) = \left(\frac{100}{100+t}\right)^2$$

Calculate ${}_{5|}q_{40}$

Solution: First express the desired probability in terms of survival functions, using equation (6.4).

$${}_{5|}q_{40} = {}_5p_{40} - {}_6p_{40} = S_{40}(5) - S_{40}(6)$$

Then express these in terms of S_0, using equation (6.2).

$$\begin{aligned}
S_{40}(5) &= \frac{S_0(45)}{S_0(40)} = \frac{(100/145)^2}{(100/140)^2} = 0.932224 \\
S_{40}(6) &= \frac{S_0(46)}{S_0(40)} = \frac{(100/146)^2}{(100/140)^2} = 0.919497
\end{aligned}$$

The answer[3] is ${}_{5|}q_{40} = 0.932224 - 0.919497 = \boxed{0.012727}$

6.3 Life tables

A **life table** is a concrete way to look at the survivorship random variable. A life table specifies a certain number of lives at a starting integer age x_0. Usually $x_0 = 0$. This number of lives at age x_0 is called the *radix*. Then for each integer $x > x_0$, the expected number of survivors is listed. The notation for the number of lives listed in the table for age x is l_x.

Assuming for simplicity that $x_0 = 0$, the random variable for the number of lives at each age, $\mathcal{L}(x)$, is a binomial random variable with parameters l_0 and ${}_xp_0$, so the expected number of lives is $l_x = l_0\,{}_xp_0$. Similarly, $l_{x+t} = l_x\,{}_tp_x$. More importantly, ${}_tp_x$ can be calculated from the table using ${}_tp_x = l_{x+t}/l_x$.

A life table also lists the expected number of deaths at each age; d_x is the notation for this concept. Thus $d_x = l_x - l_{x+1} = l_x\,q_x$. Therefore, $q_x = d_x/l_x$. The life table also makes it easy to compute ${}_{u|t}q_x = (l_{x+u} - l_{x+t+u})/l_x$.

Here's an example of a life table:

x	l_x	d_x
70	100	10
71	90	15
72	75	15
73	60	20
74	40	18

[3]Without intermediate rounding, the answer would be 0.012726. We will often show rounded values but use unrounded values in our calculations.

Notice that on each line, $d_x = l_x - l_{x+1}$. We can deduce that $l_{75} = l_{74} - d_{74} = 40 - 18 = 22$.

Example 6D Using the life table above, calculate ${}_3p_{71}$. ■

Solution:

$$ {}_3p_{71} = \frac{l_{74}}{l_{71}} = \frac{40}{90} = \boxed{\frac{4}{9}} $$ □

The notation ${}_nd_x$ is the number of deaths occurring within n years after age x; $d_x = {}_1d_x$, and ${}_nd_x = \sum_{j=0}^{n-1} d_{x+j}$.

On exams, questions are based on the Standard Ultimate Life Table. In this table, the column for d_x is omitted, and must be deduced from l_x if needed. However, a column of mortality rates, $1000q_x$, is given, even though mortality rates could also be computed from the l_x's. On pre-2018 exams, the Illustrative Life Table was used. I have translated all such old questions to the Standard Ultimate Life Table.

Since life tables are so convenient, it is sometimes easier to build a life table to solve a probability question than to work the question out directly.

Continuation of Example 6B. Redo Example 6B using life tables.

Solution: We will arbitrarily use a radix of 1,000,000 at age 60. Then we recursively calculate l_x, $x = 61, \ldots, 65$ using $l_{x+1} = l_x(1 - q_x)$.

x	q_x	l_x
60	0.001	1,000,000
61	0.002	999,000
62	0.003	997,002
63	0.004	994,011
64	0.005	990,035
65		985,085

The answer is $(997{,}002 - 985{,}085)/1{,}000{,}000 = \boxed{0.011917}$. □

An exam question may ask you to fill in the blanks in a life table using probabilities. However, this type of question is probably too simple for SOA exams.

Example 6E You are given the following life table:

x	l_x	d_x	p_x
0		50	
1			0.98
2	890		

Calculate ${}_2p_0$. ■

Solution: We back out l_0:

$$ l_1 = \frac{890}{0.98} = 908.16 $$
$$ l_0 = 908.16 + 50 = 958.16 $$

Hence ${}_2p_0 = 890/958.2 = \boxed{0.9288}$. □

Quiz 6-2 You are given:

i) $d_{48} = 80$
ii) $l_{50} = 450$
iii) ${}_{3|2}q_{45} = 1/6$
iv) ${}_{3}p_{45} = 2/3$

Determine d_{49}.

6.4 Number of survivors

Suppose you are given a group of n individuals age x. If the probability of survival to age y for each individual is p, then the number of survivors N is **binomial**, so

$$\mathbf{E}[N] = np$$
$$\text{Var}(N) = np(1-p)$$

For a large group, you can estimate probabilities and percentiles of the number of survivors using the normal approximation.

Example 6F You have a group of 100 men age 80. Their mortality follows the Standard Ultimate Life Table.

Calculate the mean and variance of the number who survive to age 90. Then use the normal approximation to calculate the 95th percentile of the number of survivors to age 90. ■

Solution: Based on the Standard Ultimate Life Table,

$$ {}_{10}p_{80} = \frac{l_{90}}{l_{80}} = \frac{41{,}841.1}{75{,}657.2} = 0.553035$$

Let N be the number of survivors to age 90. Then

$$\mathbf{E}[N] = 100(0.553035) = \boxed{55.3035}$$
$$\text{Var}(N) = 100(0.553035)(1-0.553035) = \boxed{24.7187}$$

Using the normal approximation, the 95th percentile of the number of survivors to age 90 is $55.3035 + 1.645\sqrt{24.7187} = \boxed{63.4821}$. □

Example 6G For each member of a group of 100 age 70, there is a 90% probability of standard health and a 10% probability of substandard health. For standard lives, $q_{70} = 0.01$ and for substandard lives $q_{70} = 0.02$. All lives in the group are mutually independent.

Calculate the variance of the number of survivors to age 71. ■

Solution: We use the conditional variance formula (4.1), conditioning on whether a life is standard.

Let N be the number of survivors for each individual (so N is between 0 and 1). In the following, I is the indicator variable indicating whether a person is standard or substandard. The expressions $\mathbf{E}[0.0099, 0.0196]$ and $\text{Var}(0.99, 0.98)$ mean the expected value and variance of a random variable that assumes only the two given values, with probabilities of each value deduced from the context.

$$\begin{aligned}
\mathbf{E}[N \mid \text{standard}] &= 0.99 \\
\text{Var}(N \mid \text{standard}) &= 0.99(0.01) = 0.0099 \\
\mathbf{E}[N \mid \text{substandard}] &= 0.98 \\
\text{Var}(N \mid \text{substandard}) &= 0.98(0.02) = 0.0196 \\
\text{Var}(N) &= \mathbf{E}[\text{Var}(N \mid I)] + \text{Var}(\mathbf{E}[N \mid I]) \\
&= \mathbf{E}[0.0099, 0.0196] + \text{Var}(0.99, 0.98)
\end{aligned}$$

On the following line, we calculate Var(0.99, 0.98) using the Bernoulli shortcut (Section 3.3): product of the probabilities of standard and substandard, times the square of the difference between the values.

$$\mathrm{Var}(N) = 0.9(0.0099) + 0.1(0.0196) + (0.1)(0.9)(0.99 - 0.98)^2 = 0.010879$$

The variance for the group of 100 independent lives is 100 times the variance for one life, or **1.0879** □

Note

Although we've spoken about human mortality throughout this lesson, everything applies equally well to any situation in which you want to study the time of failure random variable. Failure doesn't even have to be a bad thing. Thus, we can study random variables such as time until becoming an FSA, time until first marriage, etc. Define a random variable measuring time until the event of interest, and then you can define the actuarial functions and build a life table.

Table 6.1: Summary for this lesson

Probability notation

$$F_x(t) = \Pr(T_x \le t)$$
$$S_x(t) = 1 - F_x(t) = \Pr(T_x > t)$$
$$S_{x+u}(t) = \frac{S_x(t+u)}{S_x(u)} \tag{6.1}$$
$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} \tag{6.2}$$
$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{1 - F_0(x)} \tag{6.3}$$

Actuarial notation

${}_tp_x = S_x(t) =$ probability that (x) survives t years
${}_tq_x = F_x(t) =$ probability that (x) dies within t years
${}_{t|u}q_x =$ probability that (x) survives t years and then dies in the next u years.

$$\begin{aligned} {}_{t+u}p_x &= {}_tp_x\,{}_up_{x+t} \\ {}_{t|u}q_x &= {}_tp_x\,{}_uq_{x+t} \\ &= {}_tp_x - {}_{t+u}p_x \qquad (6.4) \\ &= {}_{t+u}q_x - {}_tq_x \qquad (6.5) \end{aligned}$$

Life table functions

l_x is the number of lives at exact age x.
d_x is the number of deaths at age x; in other words, the number of deaths between exact age x and exact age $x+1$.
${}_nd_x$ is the number of deaths between exact age x and exact age $x+n$.

$${}_tp_x = \frac{l_{x+t}}{l_x}$$
$${}_tq_x = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x}$$
$${}_{t|u}q_x = \frac{{}_ud_{x+t}}{l_x} = \frac{l_{x+t} - l_{x+t+u}}{l_x}$$

Exercises

Probability Notation

6.1. **[CAS4-S87:16]** (1 point) You are given the following survival function:

$$S_0(x) = \begin{cases} (10000 - x^2)/10000 & 0 \le x \le 100 \\ 0 & x > 100 \end{cases}$$

Calculate q_{32}.

(A) Less than 0.005
(B) At least 0.005, but less than 0.006
(C) At least 0.006, but less than 0.007
(D) At least 0.007, but less than 0.008
(E) At least 0.008

6.2. You are given

i) $S_{10}(25) = 0.9$
ii) $F_{20}(15) = 0.05$.

Determine $S_{10}(10)$.

6.3. You are given the survival function

$$S_0(t) = \frac{t^2 - 190t + 9000}{9000} \qquad t \le 90$$

Determine the probability that a life currently age 36 dies between ages 72 and 81.

Actuarial Notation

6.4. **[CAS4-S88:16]** (1 point) Which of the following are equivalent to ${}_tp_x$?

(A) ${}_{t|u}q_x - {}_{t+u}p_x$
(B) ${}_{t+u}q_x - {}_tq_x + {}_{t+u}p_x$
(C) ${}_tq_x - {}_{t+u}q_x + {}_tp_{x+u}$
(D) ${}_tq_x - {}_{t+u}q_x - {}_tp_{x+u}$
(E) The correct answer is not given by (A), (B), (C), or (D).

6.5. You are given:

i) The probability that a person age 50 is alive at age 55 is 0.9.
ii) The probability that a person age 55 is not alive at age 60 is 0.15.
iii) The probability that a person age 50 is alive at age 65 is 0.54.

Calculate the probability that a person age 55 dies between ages 60 and 65.

Exercises continue on the next page . . .

6.6. **[4-S86:13]** You are given that ${}_{t|}q_x = 0.10$ for $t = 0, 1, \ldots, 9$.

Calculate ${}_2p_{x+5}$.

(A) 0.40 (B) 0.60 (C) 0.72 (D) 0.80 (E) 0.81

6.7. **[150-82-94:10]** You are given the following:

i) The probability that a person age 20 will survive 30 years is 0.7.
ii) The probability that a person age 45 will die within 5 years and that another person age 40 will survive 5 years is 0.0475.
iii) The probability that a person age 20 will survive 20 years and that another person age 40 will die within 5 years is 0.04.

Calculate the probability that a person age 20 will survive 25 years.

(A) 0.74 (B) 0.75 (C) 0.76 (D) 0.77 (E) 0.78

6.8. **[CAS4A-S98:13]** (2 points) You are given the following information:

1. The probability that two 70-year-olds are both alive in 20 years is 16%.
2. The probability that two 80-year-olds are both alive in 20 years is 1%.
3. There is an 8% chance of a 70-year-old living 30 years.
4. All lives are independent and have the same expected mortality.

Determine the probability of an 80-year-old living 10 years.

(A) Less than 0.35
(B) At least 0.35, but less than 0.45
(C) At least 0.45, but less than 0.55
(D) At least 0.55, but less than 0.65
(E) At least 0.65

Life Tables

6.9. You are given the following mortality table:

x	l_x	d_x	${}_{x-60\|}q_{60}$
60	1000		
61		100	
62			0.07
63	780		

Calculate q_{60}.

Exercises continue on the next page . . .

6.10. **[CAS4A-S93:2]** (1 point) You are given the following information:

$$l_1 = 9700$$
$$q_1 = q_2 = 0.020$$
$$q_4 = 0.026$$
$$d_3 = 232$$

Determine the expected number of survivors to age 5.

(A) Less than 8,845
(B) At least 8,845, but less than 8,850
(C) At least 8,850, but less than 8,855
(D) At least 8,855, but less than 8,860
(E) At least 8,860

6.11. **[CAS4A-F93:1]** (1 point) You are given the following mortality table:

x	q_x	l_x	d_x
20		30,000	1,200
21			
22		27,350	
23	0.0700		
24	0.0790	23,900	

Determine the probability that a life age 21 will die within two years.

(A) Less than 0.0960
(B) At least 0.0960, but less than 0.1010
(C) At least 0.1010, but less than 0.1060
(D) At least 0.1060, but less than 0.1110
(E) At least 0.1110

6.12. Jack enters a mortality study at age 25. He dies between ages 65 and 67.

Which of the following does *not* express the likelihood of this event?

(A) ${}_{40}p_{25} \cdot {}_2q_{65}$
(B) $\dfrac{S_0(65) - S_0(67)}{S_0(25)}$
(C) ${}_{40}p_{25} - {}_{42}p_{25}$
(D) $\dfrac{d_{66} + d_{67}}{l_{25}}$
(E) ${}_{40|2}q_{25}$

Exercises continue on the next page . . .

6.13. **[CAS4A-S92:4]** (2 points) You are given the following mortality table:

x	l_x	q_x	d_x
50	1,000	0.020	
51			32
52			30
53			28
54		0.028	

In a group of 800 people age 50, determine the expected number who will die while age 54.

(A) Less than 21
(B) At least 21, but less than 24
(C) At least 24, but less than 27
(D) At least 27, but less than 30
(E) At least 30

6.14. **[CAS4A-S99:12]** (2 points) Given the following portion of a life table:

x	l_x	d_x	p_x	q_x
0	1,000	—	0.875	—
1	—	—	—	—
2	750	—	—	0.25
3	—	—	—	—
4	—	—	—	—
5	200	120	—	—
6	—	—	—	—
7	—	20	—	1.00

Determine the value of $p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot q_6$.

(A) Less than 0.055
(B) At least 0.055, but less than 0.065
(C) At least 0.065, but less than 0.075
(D) At least 0.075
(E) The answer cannot be determined from the given information.

6.15. **[3-S00:28]** For a mortality study on college students:

i) Students entered the study on their birthdays in 1963.
ii) You have no information about mortality before birthdays in 1963.
iii) Dick, who turned 20 in 1963, died between his 32^{nd} and 33^{rd} birthdays.
iv) Jane, who turned 21 in 1963, was alive on her birthday in 1998, at which time she left the study.
v) All lifetimes are independent.
vi) Likelihoods are based upon the Standard Ultimate Life Table.

Calculate the likelihood for these two students.

6.16. Mortality follows the Standard Ultimate Life Table. Jack and Jill are two independent lives of ages 25 and 30 respectively.

Calculate the probability of Jack and Jill both living to at least age 65 but not to age 90.

Exercises continue on the next page . . .

6.17. **[CAS4A-F98:15]** (2 points) Light bulbs burn out according to the following life table:

l_0	1,000,000
l_1	800,000
l_2	600,000
l_3	300,000
l_4	0

A new plant has 2,500 light bulbs. Burned out light bulbs are replaced with new light bulbs at the end of each year.

What is the expected number of new light bulbs that will be needed at the end of year 3?

(A) Less than 800
(B) At least 800, but less than 860
(C) At least 860, but less than 920
(D) At least 920, but less than 980
(E) At least 980

6.18. You are given the following life table:

x	l_x	d_x
80	5000	59
81	4941	77
82	4864	74
83	4790	80

Let X be the number of survivors to age 83 from a cohort of 5000 lives at age 80.

Calculate $\mathrm{Var}(X)$.

6.19. **[M-F05:31]** The graph of a piecewise linear survival function, $S_0(t)$, consists of 3 line segments with endpoints (0,1), (25,0.50), (75,0.40), (100,0).

Calculate $\dfrac{{}_{20|55}q_{15}}{{}_{55}q_{35}}$.

(A) 0.69 (B) 0.71 (C) 0.73 (D) 0.75 (E) 0.77

Additional old SOA Exam LTAM questions: F19:1

Multiple choice sample questions: 2.2, 2.7, 2.12, 3.4, 3.8, 3.9

Solutions

6.1. Translate q_{32} into survival functions and use using equation (6.2).

$$\begin{aligned} q_{32} &= F_{32}(1) = 1 - S_{32}(1) \\ S_{32}(1) &= \frac{S_0(33)}{S_0(32)} \\ &= \frac{10000 - 33^2}{10000 - 32^2} = 0.992758 \\ F_{32}(1) &= 1 - 0.992758 = \boxed{0.007242} \qquad \textbf{(D)} \end{aligned}$$

6.2. The probability of (10) surviving 25 years is the probability of (10) surviving 10 years and then another 15 years.

$$\begin{aligned} S_{10}(25) &= S_{10}(10)S_{20}(15) \\ 0.9 &= S_{10}(10)(1-0.05) \\ S_{10}(10) &= \frac{0.9}{0.95} = \boxed{0.9474} \end{aligned}$$

6.3. We will need $S_0(36)$, $S_0(72)$, and $S_0(81)$.

$$\begin{aligned} S_0(36) &= \frac{36^2 - 190(36) + 9000}{9000} = 0.384 \\ S_0(72) &= \frac{72^2 - 190(72) + 9000}{9000} = 0.056 \\ S_0(81) &= \frac{81^2 - 190(81) + 9000}{9000} = 0.019 \\ {}_{36|9}q_{36} &= {}_{36}p_{36} - {}_{45}p_{36} \\ &= \frac{S_0(72)}{S_0(36)} - \frac{S_0(81)}{S_0(36)} \\ &= \frac{0.056}{0.384} - \frac{0.019}{0.384} = \boxed{0.096354} \end{aligned}$$

6.4. Using ${}_{t|u}q_x = {}_tp_x - {}_{t+u}p_x$, (A) becomes ${}_tp_x - 2\,{}_{t+u}p_x$, so it doesn't work.
In (B), we note that ${}_{t+u}q_x + {}_{t+u}p_x = 1$, so it becomes $1 - {}_tq_x = {}_tp_x$, the correct answer.
(C) and (D) each have ${}_tp_{x+u}$ which is a function of survival to age $x+u$, and no other term to cancel it, so those expressions cannot possibly equal ${}_tp_x$ which only depends on survival to age x.
(B)

6.5. We need ${}_{5|5}q_{55} = {}_5p_{55} - {}_{10}p_{55}$.

$$\begin{aligned} {}_5p_{55} &= 1 - {}_5q_{55} = 1 - 0.15 = 0.85 \\ {}_{15}p_{50} &= 0.54 \\ {}_5p_{50}\,{}_{10}p_{55} &= 0.54 \\ 0.9\,{}_{10}p_{55} &= 0.54 \\ {}_{10}p_{55} &= 0.6 \end{aligned}$$

The answer is $0.85 - 0.6 = \boxed{0.25}$.

6.6. It's probably easiest to go from ${}_{t|}q_x$ to ${}_2p_{x+5}$, which is what we need, by using a life table. If we start with a radix of $l_x = 10$, then since $1/10$ of the population dies each year, $l_{x+t} = 10 - t$. Then

$$ {}_2p_{x+5} = \frac{l_{x+7}}{l_{x+5}} = \frac{10-7}{10-5} = \boxed{0.6} \qquad \textbf{(B)} $$

6.7. Perhaps the best way to do this exercise is to construct a life table. Let $l_{20} = 1$. By (i), $l_{50} = 0.7$. In (iii), we are given that the following is equal to 0.04:

$$ {}_{20}p_{20}\,{}_5q_{40} = \left(\frac{l_{40}}{l_{20}}\right)\left(\frac{l_{40} - l_{45}}{l_{40}}\right) = \frac{l_{40} - l_{45}}{l_{20}} $$

and since we set $l_{20} = 1$, we have $l_{40} - l_{45} = 0.04$. Let $x = l_{45}$, which incidentally is the final answer we're looking for, since we're looking for ${}_{25}p_{20} = l_{45}/l_{20}$ and we've set $l_{20} = 1$. The expression that is equal to 0.0475 by (ii) is

$$ {}_5q_{45}\,{}_5p_{40} = \left(\frac{l_{45} - l_{50}}{l_{45}}\right)\left(\frac{l_{45}}{l_{40}}\right) = \frac{l_{45} - l_{50}}{l_{40}} = \frac{x - 0.7}{x + 0.04} $$

Let's solve for x.

$$x - 0.7 = 0.0475(x + 0.04)$$
$$0.9525x = 0.7 + 0.04(0.0475) = 0.7019$$
$$x = \boxed{0.736903} \qquad \textbf{(A)}$$

6.8. There are three variables: $x = {}_{10}p_{70}$, $y = {}_{10}p_{80}$, and $z = {}_{10}p_{90}$. We are given

1. $(xy)^2 = 0.16 \Rightarrow xy = 0.4$
2. $(yz)^2 = 0.01 \Rightarrow yz = 0.1$
3. $xyz = 0.08$

and we want y. From the first and third statement, $z = 0.08/0.4 = 0.2$. Then from the second statement, $y = 0.1/0.2 = \boxed{0.5}$. **(C)**

6.9. Did you notice that you are given ${}_{x-60|}q_{60}$ rather than q_x?

Since ${}_{2|}q_{60} = 0.07$, then $d_{62} = 0.07l_{60} = 70$ and $l_{62} = l_{63} + d_{62} = 780 + 70 = 850$. Then $l_{61} = 850 + d_{61} = 950$ and $d_{60} = l_{60} - l_{61} = 1000 - 950 = 50$, so $q_{60} = d_{60}/l_{60} = 50/1000 = \boxed{0.05}$.

6.10. We recursively compute l_x through $x = 5$.

$$l_2 = l_1(1 - q_1) = 9700(1 - 0.020) = 9506$$
$$l_3 = l_2(1 - q_2) = 9506(1 - 0.020) = 9315.88$$
$$l_4 = l_3 - d_3 = 9315.88 - 232 = 9083.88$$
$$l_5 = l_4(1 - q_4) = 9083.88(1 - 0.026) = \boxed{8847.70} \qquad \textbf{(B)}$$

6.11. We need l_{21} and l_{23}.

$$l_{21} = 30{,}000 - 1{,}200 = 28{,}800$$
$$l_{23} = 23{,}900/(1 - 0.0700) = 25{,}698.92$$
$${}_2q_{21} = \frac{28{,}800 - 25{,}698.92}{28{,}800} = \boxed{0.1077} \qquad \textbf{(D)}$$

6.12. All of these expressions are fine except for **(D)**, which should have $d_{65} + d_{66}$ in the numerator.

6.13. $l_{54} = 1000(1 - 0.020) - 32 - 30 - 28 = 890$. Then ${}_{4|}q_{50} = (890/1000)(0.028) = 0.02492$. For 800 people, $800(0.02492) = \boxed{19.936}$. **(A)**

6.14. They gave you superfluous information for year 2. It's the usual CAS type of humor—it's not that the answer cannot be determined from the given information (almost never the right answer choice), rather there is too much information provided.

We back out $l_7 = 20$, since everyone dies that year. We calculate $l_6 = l_5 - d_5 = 80$. $l_1 = 1000(0.875) = 875$. Then what the question is asking for is

$${}_{5|}q_1 = \frac{l_6 - l_7}{l_1} = \frac{80 - 20}{875} = \boxed{0.06857} \qquad \textbf{(C)}$$

6.15. For independent lifetimes, we multiply the likelihood for each life together to get the likelihood of the joint event.

For Dick, the condition is age 20, and death occurs at age 32, so we need $\dfrac{d_{32}}{l_{20}} = \dfrac{l_{32} - l_{33}}{l_{20}}$.

For Jane, the condition is age 21 and she survived to age 56, so we need $\dfrac{l_{56}}{l_{21}}$.

Looking up the Standard Ultimate Life Table, we find

x	l_x
20	100,000.00
21	99,975.00
32	99,663.20
33	99,629.30
56	97,651.20

The answer is

$$\left(\frac{99{,}663.20 - 99{,}629.30}{100{,}000.00}\right)\left(\frac{97{,}651.20}{99{,}975.00}\right) = \boxed{0.00033112}$$

6.16. For Jack, we need $(l_{65} - l_{90})/l_{25}$, and for Jill we need $(l_{65} - l_{90})/l_{30}$. From the Standard Ultimate Life Table, we have

x	l_x
25	99,871.1
30	99,727.3
65	94,579.7
90	41,841.1

$$\left(\frac{94{,}579.7 - 41{,}841.1}{99{,}871.1}\right)\left(\frac{94{,}579.7 - 41{,}841.1}{99{,}727.3}\right) = \boxed{0.27926}$$

6.17. We have to keep track of three cohorts of light bulbs: the ones installed at the ends of years 1, 2, and 3. From the life table, the unconditional probabilities of failure are $q_0 = 0.2$ in year 0, ${}_{1|}q_0 = 0.2$ in year 1, and ${}_{2|}q_0 = 0.3$ in year 3. Of the 2500 original bulbs, 500 apiece fail in years 1 and 2 and 750 in year 3. 500 new ones are installed at the end of year 1, and from those 20% fail each subsequent year; 100 apiece fail in years 2 and 3. Finally $500 + 100 = 600$ are installed in year 2, of which 120 fail in year 3. The answer, the total number of bulbs failing in year 3, is $750 + 100 + 120 = 970$. **(D)**. Here's a table with these results:

	Number installed	Failures year 1	Failures year 2	Failures year 3
Installed initially	2500	500	500	750
Installed at end of year 1	500		100	100
Installed at end of year 2	600			120
Total failures		500	600	970

6.18. X is binomial with parameters 5000 and ${}_3p_{80} = 4790/5000$, so its variance is

$$5000\left(\frac{4790}{5000}\right)\left(1 - \frac{4790}{5000}\right) = \boxed{201.18}$$

The variance is $l_{80}\,{}_3p_{80}\,{}_3q_{80}$. Notice that the variance of the number who die in the three years is the same.

6.19. The given fraction is

$$\frac{{}_{20|55}q_{15}}{{}_{55}q_{35}} = \frac{{}_{20}p_{15}\,{}_{55}q_{35}}{{}_{55}q_{35}} = {}_{20}p_{15} = \frac{S_0(35)}{S_0(15)}$$

By linear interpolation, $S_0(15) = 0.7$ and $S_0(35) = 0.48$. So the quotient is $0.48/0.7 = \boxed{0.685714}$. **(A)**

Quiz Solutions

6-1. Use equation (6.1) to relate S_{50} to S_{40}.

$$S_{50}(30) = \frac{S_{40}(40)}{S_{40}(10)}$$
$$S_{40}(40) = 1.3 - 0.02(40) = 0.5$$
$$S_{40}(10) = 1 - 0.005(10) = 0.95$$
$$S_{50}(30) = \frac{0.5}{0.95} = \boxed{0.5263}$$

6-2. Since ${}_{3|2}q_{45} = {}_3p_{45}\,{}_2q_{48}$, we deduce ${}_2q_{48} = (1/6)/(2/3) = 1/4$, and therefore $l_{50}/l_{48} = 1 - {}_2q_{48} = 3/4$ and $l_{48} = 450(4/3) = 600$. Then $l_{50} = l_{48} - d_{48} - d_{49}$, or $450 = 600 - 80 - d_{49}$, implying $d_{49} = \boxed{70}$.

Part XIV

Practice Exams

Here are 12 practice exams to help you test your knowledge, and to pinpoint areas you are weak in so you will know what to review.

In each practice exam, the number of questions on each of the ten major topics of Exam FAM are in line with the syllabus. The questions are randomly arranged, both in terms of topic and in terms of difficulty. On real exams (or at least the ones they've released), there are some easy questions at the beginning and an easy question at the end, but these exams don't follow that rule.

Practice Exam 1

1. You are given the following observations:

2, 10, 28, 64, 100

The observations are fitted to an inverse exponential distribution using maximum likelihood.

Determine the resulting estimate of the mode.

(A) 3.2 (B) 3.4 (C) 3.6 (D) 3.8 (E) 4.0

2. For an annual premium 2-year term insurance on (60) with benefit b payable at the end of the year of death, you are given

i)

t	p_{60+t-1}
1	0.98
2	0.96

ii) The annual net premium is 25.41.

iii) $i = 0.05$.

Determine the revised annual net premium if an interest rate of $i = 0.04$ is used.

(A) 25.59 (B) 25.65 (C) 25.70 (D) 25.75 (E) 25.81

3. A reinsurance company offers a stop-loss reinsurance contract that pays the excess of annual aggregate losses over 3.

You are given:

i) Loss counts follow a binomial distribution with $m = 3$ and $q = 0.2$.

ii) Loss sizes have the following distribution:

Size	Probability
1	0.6
2	0.2
3	0.1
4	0.1

Calculate the expected annual payment under the stop-loss reinsurance contract.

(A) 0.09 (B) 0.10 (C) 0.11 (D) 0.12 (E) 0.13

4. A life age 90 is subject to mortality following Makeham's law with $A = 0.0005$, $B = 0.0008$, and $c = 1.07$.

Curtate life expectancy for this life is 6.647 years.

Using Woolhouse's formula with three terms, compute complete life expectancy for this life.

(A) 7.118 (B) 7.133 (C) 7.147 (D) 7.161 (E) 7.176

5. Your company sells whole life insurance policies. At a meeting with the Enterprise Risk Management Committee, it was agreed that you would limit the face amount of the policies sold so that the probability that the present value of the benefit at issue is greater than 1,000,000 is never more than 0.05.

You are given:

i) The insurance policies pay a benefit equal to the face amount b at the moment of death.
ii) The force of mortality is $\mu_x = 0.001(1.05^x)$, $x > 0$
iii) $\delta = 0.06$

Determine the largest face amount b for a policy sold to a purchaser who is age 45.

(A) 1,350,000 (B) 1,400,000 (C) 1,450,000 (D) 1,500,000 (E) 1,550,000

6. Endowment insurance is no longer offered by major insurers in North America or the UK.

Consider the following reasons:

I. The product has low returns.
II. The product is not flexible.
III. There are onerous tax provisions on the product.

State which of these reasons is given in *Actuarial Mathematics for Life Contingent Risks*.

(A) None (B) I and II only (C) I and III only (D) II and III only
(E) The correct answer is not given by (A) , (B) , (C) , or (D) .

7. A rate filing for six-month policies will be effective starting October 1, CY6 for 2 years.

Losses for this rate filing were incurred in AY1 and the amount paid through 12/31/AY4 is 3,500,000.

Trend is at annual effective rate of 6.5%.

Loss development factors are:

$$3/2\text{: } 1.50, \qquad 4/3\text{: } 1.05, \qquad \infty/4\text{: } 1.05$$

Calculate trended and developed losses for AY1.

(A) Less than 5,600,000
(B) At least 5,600,000, but less than 5,700,000
(C) At least 5,700,000, but less than 5,800,000
(D) At least 5,800,000, but less than 5,900,000
(E) At least 5,900,000

8. You are given that $\mu_x = 0.002x + 0.005$.

Calculate ${}_{5|}q_{20}$.

(A) 0.015 (B) 0.026 (C) 0.034 (D) 0.042 (E) 0.050

9. A life age 60 is subject to Gompertz's law with $B = 0.001$ and $c = 1.05$.

Calculate $e_{60:\overline{2}|}$ for this life.

(A) 1.923 (B) 1.928 (C) 1.933 (D) 1.938 (E) 1.943

Exam questions continue on the next page . . .

10. For a fully discrete 20-year deferred whole life insurance of 1000 on (50), you are given:

i) Premiums are payable for 20 years.
ii) The net premium is 12.
iii) Deaths are uniformly distributed between integral ages.
iv) $i = 0.1$
v) ${}_9V = 240$ and ${}_{9.5}V = 266.70$.

Calculate ${}_{10}V$, the net premium reserve at the end of year 10.

(A) 272.75 (B) 280.00 (C) 281.40 (D) 282.28 (E) 282.86

11. You are given:

i) Z_1 is the present value random variable for a 10-year term insurance paying 1 at the moment of death of (45).
ii) Z_2 is the present value random variable for a 20-year deferred whole life insurance paying 1 at the moment of death of (45).
iii) $\mu = 0.02$
iv) $\delta = 0.04$

Calculate $\mathrm{Cov}(Z_1, Z_2)$.

(A) −0.042 (B) −0.028 (C) −0.023 (D) −0.015 (E) −0.009

12. On an automobile liability coverage, annual claim counts follow a negative binomial distribution with mean 0.2 and variance 0.3. Claim sizes follow a two-parameter Pareto distribution with $\alpha = 3$ and $\theta = 10$. Claim counts and claim sizes are independent.

Calculate the variance of annual aggregate claim costs.

(A) 22.5 (B) 25.0 (C) 27.5 (D) 32.5 (E) 35.0

13. For a fully continuous whole life insurance of 1000 on (x):

i) The gross premium is paid at an annual rate of 25.
ii) The variance of future loss is 500,000.
iii) $\delta = 0.06$

Employees are able to obtain this insurance for a 20% discount.

Determine the variance of future loss for insurance sold to employees.

(A) 320,383 (B) 301,261 (C) 442,907 (D) 444,444 (E) 456,253

14. An excess of loss catastrophe reinsurance treaty covers the following layers, expressed in millions:

80% of 100 excess of 100
85% of 200 excess of 200
90% of 400 excess of 400

Calculate the reinsurance payment for a catastrophic loss of 650 million.

(A) 225 million (B) 475 million (C) 495 million (D) 553 million (E) 585 million

Exam questions continue on the next page . . .

15. A study on claim sizes produced the following results:

Claim size	Number	Deductible	Limit
500	4	None	10,000
1000	4	500	None
2000	3	500	None
5000	2	None	10,000
At limit	5	None	10,000

A single-parameter Pareto with $\theta = 400$ is fitted to the data using maximum likelihood.

Determine the estimate of α.

(A) 0.43 (B) 0.44 (C) 0.45 (D) 0.61 (E) 0.62

16. You are given the following statements regarding disability insurance.

I. "Own job" insurance tends to be cheaper than "any job" insurance.

II. For a policy with benefit period to 65, longer off periods make the insurance more expensive.

III. The cost of a policy increases as the benefit period increases.

(A) None of I, II, or III is true
(B) I and II only
(C) I and III only
(D) II and III only
(E) The answer is not given by (A), (B), (C), or (D)

17. Losses on an insurance coverage follow a distribution with density function

$$f(x) = \frac{3}{100^3}(100 - x)^2 \qquad 0 \le x \le 100$$

Losses are subject to an ordinary deductible of 15.

Calculate the loss elimination ratio.

(A) 0.46 (B) 0.48 (C) 0.50 (D) 0.52 (E) 0.54

18. You own 100 shares of a stock whose current price is 42. You would like to hedge your downside exposure by buying 100 6-month European put options with a strike price of 40. You are given:

i) The Black-Scholes-Merton framework is assumed.
ii) The continuously compounded risk-free interest rate is 5%.
iii) The stock pays no dividends.
iv) The stock follows a lognormal process with $\mu = 0.06$ and $\sigma = 0.22$.

Determine the cost of the put options.

(A) 121 (B) 123 (C) 125 (D) 127 (E) 129

Exam questions continue on the next page . . .

19. For a health insurance coverage, there are two types of policyholders.

75% of policyholders are healthy. Annual claim costs for those policyholders have mean 2,000 and variance 10,000,000.

25% of policyholders are in bad health. Annual claim costs for those policyholders have mean 10,000 and variance 50,000,000.

Calculate the variance of annual claim costs for a policyholder selected at random.

(A) 20,000,000 (B) 28,000,000 (C) 32,000,000 (D) 44,000,000 (E) 48,000,000

20. For loss size X, you are given:

x	$\mathbf{E}[X \wedge x]$
1000	400
2000	700
3000	900
4000	1000
5000	1100
∞	2500

An insurance coverage has an ordinary deductible of 2000.

Calculate the loss elimination ratio after 100% inflation if the deductible is not changed.

(A) 0.08 (B) 0.14 (C) 0.16 (D) 0.20 (E) 0.40

21. A special 9-year term insurance on (x) pays the following benefit at the end of the year of death:

Year of death t	1	2	3	4	5	6	7	8	9
Benefit b_t	1	2	3	4	5	4	3	2	1

$(DA)^{1}_{x:\overline{n}|}$ denotes the expected present value of a decreasing term insurance that pays a benefit of $n+1-k$ at the end of the year if death occurs in year k, $1 \le k \le n$.

You are given the following expected present values for increasing and decreasing term insurances:

n	$(IA)^{1}_{x:\overline{n}|}$	$(DA)^{1}_{x:\overline{n}|}$
4	0.593	0.628
5	0.848	0.923
9	1.970	2.513
10	2.219	2.986

Determine the expected present value of the special term insurance.

(A) 0.7 (B) 0.8 (C) 1.3 (D) 1.4 (E) 1.8

Exam questions continue on the next page . . .

22. In a study on loss sizes on automobile liability coverage, you are given:

i) 5 observations $x_1, \ldots, x_5$ from a plan with no deductible and a policy limit of 10,000.
ii) 5 observations at the limit from a plan with no deductible and a policy limit of 10,000.
iii) 5 observations $y_1, \ldots, y_5$ from a plan with a deductible of 10,000 and no policy limit.

Which of the following is the likelihood function for this set of observations?

(A) $\prod_{i=1}^{5} f(x_i) \prod_{i-1}^{5} f(y_i)$

(B) $\bigl(1 - F(10{,}000)\bigr)^5 \prod_{i=1}^{5} f(x_i) \prod_{i-1}^{5} f(y_i)$

(C) $\dfrac{\bigl(1 - F(10{,}000)\bigr)^5 \prod_{i=1}^{5} f(x_i) \prod_{i-1}^{5} f(y_i)}{\bigl(F(10{,}000)\bigr)^5}$

(D) $\dfrac{\prod_{i=1}^{5} f(x_i) \prod_{i=1}^{5} f(y_i)}{\bigl(1 - F(10{,}000)\bigr)^5}$

(E) $\dfrac{\bigl(F(10{,}000)\bigr)^5 \prod_{i=1}^{5} f(x_i) \prod_{i-1}^{5} f(y_i)}{\bigl(1 - F(10{,}000)\bigr)^5}$

23. For a fully discrete 25-year term life insurance on (45) with face amount 200,000, you are given:

i) Mortality follows the Standard Ultimate Life Table.
ii) Deaths are uniformly distributed between integer years.
iii) Gross premium payable quarterly is 130.
iv) Per premium and per policy expenses are

	Percent of Premium	Per Policy
First year	60%	250
Renewal	5%	30

v) Per premium expenses are payable when premiums are payable.
vi) Per policy expenses are payable at the beginning of each year.
vii) Cost of settling a death claim is 100.
viii) $i = 0.05$.

Calculate the gross premium reserve at time 15.

(A) 4817 (B) 4822 (C) 4826 (D) 4892 (E) 4896

24. At a company, the number of sick days taken by each employee in a year follows a Poisson distribution with mean λ. Over all employees, the distribution of λ has the following density function:

$$f(\lambda) = \frac{\lambda e^{-\lambda/3}}{9}$$

Calculate the probability that an employee selected at random will take more than 2 sick days in a year.

(A) 0.59 (B) 0.62 (C) 0.66 (D) 0.70 (E) 0.74

Exam questions continue on the next page . . .

25. A minor medical insurance coverage has the following provisions:

i) Annual losses in excess of 10,000 are not covered by the insurance.
ii) The policyholder pays the first 1,000 of annual losses.
iii) The insurance company pays 60% of the excess of annual losses over 1,000, after taking into account the limitation mentioned in (i).

Annual losses follow a two-parameter Pareto distribution with $\alpha = 4$ and $\theta = 8000$.

Calculate expected annual payments for one policyholder under this insurance.

(A) 983 (B) 1004 (C) 1025 (D) 1046 (E) 1067

26. Annual claim frequency follows a Poisson distribution. Loss sizes follow a Weibull distribution with $\tau = 0.5$.

Full credibility for aggregate loss experience is granted if the probability that aggregate losses differ from expected by less than 6% is 95%.

Determine the number of expected claims needed for full credibility.

(A) 6403 (B) 6755 (C) 7102 (D) 7470 (E) 7808

27. For a mortality table, you are given

i) Uniform distribution of deaths is assumed between integral ages.
ii) $\mu_{30.25} = 1$
iii) $\mu_{30.5} = \frac{4}{3}$

Determine $\mu_{30.75}$.

(A) $\frac{5}{3}$ (B) 2 (C) $\frac{7}{3}$ (D) $\frac{5}{2}$ (E) 3

28. Let X be the random variable with distribution function

$$F_X(x) = 1 - 0.6e^{-x/10} - 0.4e^{-x/20}$$

Calculate $\mathrm{TVaR}_{0.95}(X)$.

(A) 59 (B) 60 (C) 61 (D) 62 (E) 63

29. You are given:

i) For a cohort of 100 newly born children, the force of mortality is constant and equal to 0.01.
ii) Birthday cards are sent each year to all lives in the cohort beginning on their 80^{th} birthdays, for as long as they live.

Determine the expected number of birthday cards each member of this cohort receives.

(A) 44.7 (B) 44.9 (C) 45.2 (D) 45.5 (E) 45.7

30. A company has 100 shares of ABC stock. The current price of ABC stock is 30. ABC stock pays no dividends.

The company would like to guarantee its ability to sell the stock at the end of six months for at least 28.

European call options on the same stock expiring in 6 months with exercise price 28 are available for 4.10.

The continuously compounded risk-free interest rate is 5%.

Determine the cost of the hedge.

(A) 73 (B) 85 (C) 99 (D) 126 (E) 141

Exam questions continue on the next page . . .

31. You are given that $A_x = 0.4 + 0.01x$ for $x < 60$.

A fully discrete whole life insurance on (30) pays a benefit of 1 at the end of the year of death.

Calculate the net premium reserve at time 20 for this insurance.

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

32. For a temporary life annuity-due of 1 per year on (30), you are given:

i) The annuity makes 20 certain payments.
ii) The annuity will not make more than 40 payments.
iii) Mortality follows the Standard Ultimate Life Table.
iv) $i = 0.05$

Determine the expected present value of the annuity.

(A) 17.79 (B) 17.83 (C) 17.87 (D) 17.91 (E) 17.95

33. You are given

Accident Year	Cumulative Payments through Development Year 0	1	2	Earned Premium
AY1	25,000	41,000	48,000	120,000
AY2	30,000	45,000		140,000
AY3	33,000			150,000

The loss ratio is 60%.

Calculate the loss reserve using the loss ratio method.

(A) 100,000 (B) 105,000 (C) 110,000 (D) 115,000 (E) 120,000

34. You are given:

Accident Year	Incurred Losses through Development Year 0	1	2	3	4	Earned Premium
AY1	7,800	8,900	9,500	11,000	11,000	16,000
AY2	9,100	9,800	10,500	10,800		20,000
AY3	8,600	9,500	10,100			23,000
AY4	9,500	10,000				24,000
AY5	10,700					25,000

The expected loss ratio is 0.7.

Losses mature at the end of 3 years.

Calculate the IBNR reserve using the Bornhuetter-Ferguson method with volume-weighted average loss development factors.

(A) 7,100 (B) 7,200 (C) 7,300 (D) 7,400 (E) 7,500

Solutions to the above questions begin on page 1445.

Appendices

Appendix A. Solutions to the Practice Exams

Answer Key for Practice Exam 1

1	D
2	C
3	E
4	A
5	A
6	B
7	D
8	D
9	E
10	D
11	D
12	A
13	C
14	B
15	C
16	D
17	B
18	E
19	C
20	C
21	C
22	A
23	E
24	E
25	A
26	A
27	B
28	E
29	C
30	E

Practice Exam 1

1. [**Lesson** 63] The likelihood function, ignoring the multiplicative constant $1/\prod x_i^2$, is

$$L(\theta) = \theta^5 e^{-\theta \sum 1/x_i}$$

and logging and differentiating,

$$l(\theta) = 5\ln\theta - \theta\sum\frac{1}{x_i}$$
$$\frac{\mathrm{d}l}{\mathrm{d}\theta} = \frac{5}{\theta} - \sum\frac{1}{x_i} = 0$$