Solutions to EA-1 Examination
Spring, 2004

Question 1

The yield rate of the bond, and the reinvestment interest rate for the coupons received must be converted to semi-annual rates of interest to match the coupon frequency.

At the 5% effective annual yield,

$$i^{(2)}/2 = 1.05^{1/2} - 1 = .024695, \text{ or } 2.4695\%$$

At 6% per year convertible quarterly (1.5% yield per quarter),

$$j^{(2)}/2 = 1.015^{2} - 1 = .030225, \text{ or } 3.0225\%$$

The purchase price of a bond is equal to the present value of the maturity value plus the present value of the coupons. The coupons of the given bond are paid semi-annually in the amount of \$30 every six months (6% of \$1,000 divided by two). Using the semiannual yield rate of 2.4695%,

Purchase price of the bond = $1{,}000v^{40} + 30\,a_{\overline{40}|.024695} = 377 + 757 = 1{,}134$

The accumulated value for Smith of the redemption payment of 1,000 and the reinvested coupons after 20 years (accumulated using a semiannual interest rate of 3.0225%) is:

Accumulated value = $1{,}000 + 30\,s_{\overline{40}|.030225} = 3{,}274$

The equation of value can be written as:

$1{,}134 \times (1 + i)^{20} = 3{,}274$
$(1 + i)^{20} = 2.8871$

Using logarithms, $\ln(1 + i)^{20} = \ln(2.8871) \Rightarrow 20\ln(1 + i) = 1.0603$
$\ln(1 + i) = .0530 \Rightarrow 1 + i = e^{.0530}$
$i = .0544$, or 5.44%

Answer is D.

Question 2

The three annuity payments will be made on 7/1/2009, 1/1/2010, and 7/1/2010. Since the force of interest changes each year, the annual rate of interest also changes each year. The rate of interest must be determined for each year from 2004 through 2011. Note that the force of interest can be thought of as an annual rate of interest convertible over a large number of periods (say 10,000, for this question).

For 2004, $\delta_0 = 1/50 = .020000 \quad \Rightarrow \quad$ assume $i^{(10,000)} = .02$

$$\Rightarrow \quad i_0 = (1 + \frac{.02}{10,000})^{10,000} - 1 = .0202$$

For 2005, $\delta_1 = 1/52 = .019231 \quad \Rightarrow \quad$ assume $i^{(10,000)} = .019231$

$$\Rightarrow \quad i_1 = (1 + \frac{.019231}{10,000})^{10,000} - 1 = .019417$$

For 2006, $\delta_2 = 1/54 = .018519 \quad \Rightarrow \quad$ assume $i^{(10,000)} = .\ 018519$

$$\Rightarrow \quad i_2 = (1 + \frac{.018519}{10,000})^{10,000} - 1 = .018692$$

For 2007, $\delta_3 = 1/56 = .017857 \quad \Rightarrow \quad$ assume $i^{(10,000)} = .\ 017857$

$$\Rightarrow \quad i_3 = (1 + \frac{.017857}{10,000})^{10,000} - 1 = .018017$$

For 2008, $\delta_4 = 1/58 = .017241 \quad \Rightarrow \quad$ assume $i^{(10,000)} = .\ 017241$

$$\Rightarrow \quad i_4 = (1 + \frac{.017241}{10,000})^{10,000} - 1 = .017390$$

For 2009, $\delta_5 = 1/60 = .016667 \quad \Rightarrow \quad$ assume $i^{(10,000)} = .\ 016667$

$$\Rightarrow \quad i_5 = (1 + \frac{.016667}{10,000})^{10,000} - 1 = .016807$$

For 2010, $\delta_6 = 1/62 = .016129 \quad \Rightarrow \quad$ assume $i^{(10,000)} = .\ 016129$

$$\Rightarrow \quad i_5 = (1 + \frac{.016129}{10,000})^{10,000} - 1 = .016260$$

For the annuity payments made on July 1, a semiannual rate of interest is needed.

For 2009, $i^{(2)}/2 = 1.016807^{1/2} - 1 = .008368$
For 2010, $i^{(2)}/2 = 1.016260^{1/2} - 1 = .008097$

Present value of annuity on 1/1/2009

$$= (500/1.008368) + (500/1.016807) + (500/(1.016807 \times 1.008097)) = 1,475$$

Present value of annuity on 1/1/2004

$$= 1,475/(1.0202 \times 1.019417 \times 1.018692 \times 1.018017 \times 1.017390) = 1,345$$

Answer is A.

Question 3

The present value of this annuity can be written as:

$$PV = (Ia)_{\overline{10|}} + (Da)_{\overline{9|}}\, v^{10} = \frac{\ddot{a}_{\overline{10|}} - 10v^{10}}{.05} + \frac{9 - a_{\overline{9|}}}{.05}\, v^{10} = 39.37 + 23.23 = 62.60$$

Answer is C.

Question 4

The purchase price need to yield at least 5% would be equal to the price based upon the call date that would yield the lowest purchase price. Since the coupon rate exceeds the yield rate (and the investment will be redeemed at par value – the face value of the bond), it is to the purchaser's advantage for the coupon payments to continue as long as possible (as the yield is higher than the desired 5%). The worst case for the purchaser would be if the bond is called at the earliest date (1/1/2014). The purchase price of a bond is equal to the present value of the maturity value plus the present value of the coupons. The purchase price if the bond is called on 1/1/2014 is:

Purchase price of the bond $= 1{,}000v^{10} + 60\, a_{\overline{10|}} = 614 + 463 = 1{,}077$

Answer is A.

Question 5

The payments under the terms of the annuity are guaranteed for the first 10 years, and the payments after the first 10 years are made only if the annuitant was still alive 10 years earlier. Using first principles to write an expression for the present value of the annuity:

$$\begin{aligned}
PV &= 1{,}000\, a_{\overline{10|}} + 1{,}000(v^{11}\, p_{65} + v^{12}\, {}_2p_{65} + v^{13}\, {}_3p_{65} + \ldots) \\
&= 1{,}000\, a_{\overline{10|}} + 1{,}000v^{10}(v\, p_{65} + v^{2}\, {}_2p_{65} + v^{3}\, {}_3p_{65} + \ldots) \\
&= 1{,}000\, a_{\overline{10|}} + 1{,}000v^{10}a_{65} \\
&= 7{,}722 + 6{,}247 \\
&= 13{,}969
\end{aligned}$$

Answer is D.

Question 6

Since P is the monthly payment, 12P represents the annual payment.

Items (ii) and (iii) in the actuarially equivalent annuity are reversionary annuities. Recall that a reversionary annuity can be written as the difference between a life annuity to the annuitant, and a joint annuity payable while both the annuitant and the beneficiary are alive.

Setting the value of the actuarially equivalent benefits equal to each other:

$$1{,}000 \times 12\,\ddot{a}^{(12)}_{65} = 12P\,\ddot{a}^{(12)}_{63:65} + 12P/2(\ddot{a}^{(12)}_{63} - \ddot{a}^{(12)}_{63:65}) + (12 \times 1{,}000)(\ddot{a}^{(12)}_{65} - \ddot{a}^{(12)}_{63:65})$$

$11{,}500 = 9P + 1.55P + 2{,}500$

$P = 853$

Answer is D.

Question 7

The outstanding balance of a loan is equal to the present value of the remaining payments. The outstanding balance of this loan immediately after the 6^{th} payment (including the additional \$10,000 payment) is:

Outstanding balance = $5{,}000\,a_{\overline{4|}} - 10{,}000 = 16{,}561 - 10{,}000 = 6{,}561$

The present value of the remaining annual payments of \$1,000 must be equal to \$6,561. The remaining period can be determined.

$$1{,}000\,a_{\overline{n|}} = 6{,}561 \Rightarrow a_{\overline{n|}} = 6.561 \Rightarrow (1 - v^n)/.08 = 6.561$$
$$\Rightarrow v^n = .47512 \Rightarrow \ln(v^n) = \ln(.47512)$$
$$\Rightarrow n \times \ln(v) = \ln(.47512) \Rightarrow n = 9.67$$

Therefore, there are 9 full payments of \$1,000 and one final payment at the end of the 10^{th} year. This can be described by the following equation:

$$6{,}561 = 1{,}000\,a_{\overline{9|}} + Xv^{10} \Rightarrow X = \$678$$

Answer is E.

Question 8

For $i = .08$, $i^{(12)}/12 = 1.08^{1/12} - 1 = .006434$

The annuity described can be looked at as an annuity with monthly payments of X for 15 years (180 payments), and additional monthly payments of \$300 for years 8 through 15 (96 payments). The equation of value is:

$$20{,}600 = X\, a_{\overline{180}|.006434} + 300\, a_{\overline{96}|.006434}\, v^{7}_{.08}$$

$X = 76.04$

Answer is C.

Question 9

The total principal repaid is equal to the principal payments of \$500 multiplied by the number of payments (20).

Principal repaid = \$500 × 20 = \$10,000

Note that the principal repaid is exactly equal to the original loan balance.

At 5% interest, the interest in the first annual payment is \$500 (5% of \$10,000). The interest in the second annual payment is \$475 (5% of \$9,500). The interest in the third annual payment is \$450 (5% of \$9,000). A pattern can be seen in the interest payments.

The total of the interest payments is:

$$\begin{aligned}\text{Total interest paid} &= 500 + 475 + 450 + \ldots + 25 \\ &= 25 \times (20 + 19 + \ldots 1) \\ &= 25 \times [(20 \times 21)/2] \\ &= 5{,}250\end{aligned}$$

Total of interest and principal = 5,250 + 10,000 = 15,250

Answer is B.

Note: Recall the formula for a series of consecutive integers:

$1 + 2 + 3 + \ldots + n = [n \times (n+1)]/2$

Question 10

The time-weighted rate of return is:

$$(1+i) = \frac{\text{Market value on 4/1/2004*}}{\text{Market value on 1/1/2004**}} \times \frac{\text{Market value on 8/1/2004*}}{\text{Market value on 4/1/2004**}} \times \frac{\text{Market value on 12/31/2004*}}{\text{Market value on 8/1/2004**}}$$

$= (85{,}000/100{,}000) \times (100{,}000/115{,}000) \times (80{,}000/80{,}000)$
$= .7391$
$A = i = -26.09\%$

* Before contributions or withdrawals
** After contributions or withdrawals

The dollar-weighted rate of return for the fund is:

$$\text{Ending Balance} = (\text{Beginning Balance})(1+j) + (\text{Contribution})(1+\tfrac{9}{12}j) - (\text{Withdrawal})(1+\tfrac{5}{12}j)$$

$$80{,}000 = (100{,}000)(1+j) + (30{,}000)(1+\tfrac{9}{12}j) - (20{,}000)(1+\tfrac{5}{12}j)$$

$B = j = -26.28\%$

$|A + B| = 26.09\% + 26.28\% = 52.37\%$

Answer is D.

Question 11

Recall that $L_x = (l_x + l_{x+1})/2$ and $m_x = d_x/(l_x - \frac{1}{2}d_x)$

$L_x = (l_x + l_{x+1})/2 \quad \Rightarrow \quad 975 = (l_x + 960)/2 \quad \Rightarrow \quad l_x = 990$

$d_x = l_x - l_{x+1} \quad \Rightarrow \quad d_x = 990 - 960 = 30$

$1{,}000m_x = 1{,}000d_x/(l_x - \frac{1}{2}d_x) = 1{,}000 \times [30/(990 - \frac{1}{2}(30))] = 30.77$

Answer is C.

Question 12

The frequency of the annuities can be ignored in this question since the value of the annuities are actuarially equivalent to each other. When the annuities are equated, the ratio of the payments divide out. The annuities can be written as follows (where x is the annuitant, and y is the survivor):

(i) $100a_x + 50(a_y - a_{xy})$
(ii) $110a_{xy} + 55(a_x - a_{xy}) + 55(a_y - a_{xy}) = 55a_x + 55a_y$
(iii) $Pa_x + .5P(a_y - a_{xy}) + .1P(a_x - a_{xy})$

Setting annuities (i) and (ii) equal to each other:

$$100a_x + 50(a_y - a_{xy}) = 55a_x + 55a_y \quad \Rightarrow \quad 45a_x - 5a_y = 50a_{xy}$$
$$\Rightarrow \quad a_{xy} = .9a_x - .1a_y$$

Setting annuities (i) and (iii) equal to each other:

$$100a_x + 50(a_y - a_{xy}) = Pa_x + .5P(a_y - a_{xy}) + .1P(a_x - a_{xy})$$
$$\Rightarrow \quad 100a_x + 50(a_y - .9a_x + .1a_y)$$
$$= Pa_x + .5P(a_y - .9a_x + .1a_y) + .1P(a_x - .9a_x + .1a_y)$$
$$\Rightarrow \quad 55a_x + 55a_y = .56Pa_x + .56Pa_y$$
$$\Rightarrow \quad P = 98.21$$

Answer is C.

Question 13

Equating the actuarially equivalent benefits,

$$30{,}000\,\ddot{a}_{62} = 50{,}000 + {}_3p_{62}\, v^3 (X\,\ddot{a}_{\overline{5|}})$$
$${}_3p_{62} = 0.99^3 = .970229$$

Substituting into the initial equation,

$$30{,}000 \times 12.67977 = 50{,}000 + (.970229 \times .816298 \times 4.387211)X$$
$$X = 95{,}080$$

Answer is E.

Question 14

For $i = .07$, $i^{(12)}/12 = 1.07^{1/12} - 1 = .005654$

$\ddot{a}^{(12)}_{\overline{10|}} = \frac{1}{12}\ddot{a}_{\overline{120|}.005654} = 7.287120$

${}_{10|}\ddot{a}^{(12)}_{62} = \ddot{a}^{(12)}_{62} - \ddot{a}^{(12)}_{62:\overline{10|}} = 9.61521 - 6.94029 = 2.67492$

Setting the two actuarially equivalent benefits equal,

$12X\ddot{a}^{(12)}_{62} = 12 \times 3{,}000 \times (\ddot{a}^{(12)}_{\overline{10|}} + {}_{10|}\ddot{a}^{(12)}_{62})$

$(12 \times 9.61521)X = 12 \times 3{,}000 \times (7.287120 + 2.67492)$

$X = 3{,}108$

Answer is C.

Question 15

At 7% per year, $\ddot{a}_{\overline{10|}} = 7.51523$

$\ddot{a}_{61:\overline{5|}} = 1 + a_{61:\overline{4|}} = 1 + vp_{61}\ddot{a}_{62:\overline{4|}} = 4.32604$

${}_{10|}\ddot{a}_{60} = \ddot{a}_{60} - \ddot{a}_{60:\overline{10|}} = 4.26982$

Setting the present value of Jones' annuity equal to 4 times the present value of Smith's annuity,

$4X\ddot{a}_{61:\overline{5|}} = 20{,}000 \times (\ddot{a}_{\overline{10|}} + {}_{10|}\ddot{a}_{60}) \Rightarrow (4 \times 4.32604)X = 20{,}000 \times (7.51523 + 4.26982)$

$\Rightarrow X = 13{,}621$

Answer is C.

Question 16

The annuity payable under annuity #1 on 12/31/2005 is: $1{,}000 \times 1.01^7 = 1{,}072.14$
The annuity payable under annuity #2 on 12/31/2005 is: $X - 25$

The annual rate of interest must be converted to a quarterly rate of interest:

For $i = .07$, $i^{(4)}/4 = 1.07^{1/4} - 1 = .017059$

As of 1/1/2006, there are 32 payments remaining under annuity #1. The present value of the future payments of annuity #1 on 1/1/2006, using the quarterly interest rate of 1.7059%, is:

$$\begin{aligned} PV &= 1{,}072.14 \times (1.01v + (1.01v)^2 + \ldots + (1.01v)^{32}) \\ &= 1{,}072.14 \times a_{\overline{32}|.006989} \\ &= 30{,}650 \end{aligned}$$

Note that $1.01v = 1.01/1.017059 = 1/(1.017059/1.01) = 1/1.006989$
This results in an implicit interest rate of .6989%.

As of 1/1/2006, there are 13 payments remaining under annuity #2. They can be thought of as 13 level payments of (X – 25), reduced by an increasing annuity of 25 for each payment (since the annuity actually reduces by a payment of 25 on each payment date). The present value of the future payments of annuity #2 on 1/1/2006, using the annual interest rate of 7%, is:

$$\begin{aligned} PV &= (X - 25)\, a_{\overline{13}|} - 25\,(Ia)_{\overline{13}|} \\ &= (X - 25)\, a_{\overline{13}|} - 25\, \frac{\ddot{a}_{\overline{13}|} - 13v^{13}}{.07} \\ &= 8.357651X - 1{,}476.14 \end{aligned}$$

Setting the two present values equal (since the two annuities have the same present value),

$$30{,}650 = 8.357651X - 1{,}476.14$$
$$X = 3{,}844$$

Answer is B.

Question 17

The sinking fund must accumulate to a amount large enough to pay off the $100,000 loan with interest. Since this occurs on 12/31/2019, the balance in the sinking fund on that date (after 16 years) is:

$100{,}000 \times 1.05^{16} = 218{,}287$

The outstanding balance of the loan as of 1/1/2011 (after 7 years) is:

$100{,}000 \times 1.05^{7} = 140{,}710$

The interest accrued on the loan in 2011 is equal to 5% of the outstanding balance.

2011 interest accrued = $140{,}710 \times .05 = 7{,}035$

The balance in the sinking as of 1/1/2011 (after 7 years of payments of $10,000 at an interest rate of i) is:

$10{,}000\, s_{\overline{7}|i} \times i$

Since the loan interest and the sinking fund interest in 2011 are equal,

$$
\begin{aligned}
7{,}035 = 10{,}000\, s_{\overline{7}|i} \times i \quad &\Rightarrow \quad 7{,}035 = 10{,}000 \frac{(1+i)^7 - 1}{i} \times i \\
&\Rightarrow \quad 7{,}035 = 10{,}000 \times [(1+i)^7 - 1] \\
&\Rightarrow \quad (1+i)^7 = 1.7035 \\
&\Rightarrow \quad \ln(1+i)^7 = \ln(1.7035) \\
&\Rightarrow \quad 7\ln(1+i) = .53268 \\
&\Rightarrow \quad \ln(1+i) = .07610 \\
&\Rightarrow \quad 1 + i = e^{.07610} \\
&\Rightarrow \quad i = .07907
\end{aligned}
$$

The value of the sinking fund on 12/31/2011 (using the interest rate of 7.907%) is:

PV of sinking fund on 12/31/2011 = $10{,}000\, s_{\overline{8}|.07907} = 106{,}010$

This amount, along with the additional 8 payments of $5,000, must accumulate at interest rate k% to 218,287 on 12/31/2019. This is represented by:

$218{,}287 = (106{,}010 \times (1+k)^8) + 5{,}000\, s_{\overline{8}|k}$

Evaluating the right side of the equation using 6.1% as k (the lowest interest rate in the available answer ranges),

$(106{,}010 \times (1.061)^8) + 5{,}000\, s_{\overline{8}|.061} = 219{,}907$

Evaluating the right side of the equation using 5.6% as k (6.1% less the difference in the range bands),

$(106{,}010 \times (1.056)^8) + 5{,}000\, s_{\overline{8}|.056} = 212{,}712$

Interpolating,

$$k = 6.1\% - (.5\% \times \frac{219{,}907 - 218{,}287}{219{,}907 - 212{,}712}) = 5.987\%$$

Answer is A.

Question 18

Based upon the given mortality for purposes of the premium,

$l_{50} = 100 - 50 = 50$ and $d_x = 1$ for each age x.

The premium for the 3-year term insurance is:

$$\begin{aligned} \text{Premium} &= 1{,}300 \text{ lives} \times [10{,}000 \times (vq_{50} + v^2\ {}_{1|}q_{50} + v^3\ {}_{2|}q_{50})] \\ &= 13{,}000{,}000 \times (\tfrac{1}{50} v + \tfrac{1}{50} v^2 + \tfrac{1}{50} v^3) \\ &= 260{,}000 \times a_{\overline{3}|.05} \\ &= 708{,}044 \end{aligned}$$

Since the actual experience for l_x is linear, the same number of lives die each year. At age 50, there are 52 lives remaining in the mortality table (102 – 50). The number of deaths each year in the actual population of 1,300 lives is 25 (1,300 ÷ 52).

The balance in the fund after 3 years is equal to the difference between the accumulated premium (paid in a single sum at the beginning of the 3 years) and the accumulated value of the death benefits paid. All accumulations are at the actual earnings experienced of 5.25%.

$Y = (708{,}044 \times 1.0525^3) - (25 \times 10{,}000 \times s_{\overline{3}|.0525}) = 825{,}518 - 790{,}064 = 35{,}454$

Answer is E.

Question 19

The present value of the joint annuity is:

$$PV = 10{,}000\,\ddot{a}_{64:65} + 6{,}000(\ddot{a}_{64} - \ddot{a}_{64:65}) + 6{,}000(\ddot{a}_{65} - \ddot{a}_{64:65})$$
$$= 6{,}000\,\ddot{a}_{64} + 6{,}000\,\ddot{a}_{65} - 2{,}000\,\ddot{a}_{64:65}$$

$\ddot{a}_{65} = 1 + a_{65} = 1 + 9.8207 = 10.8207$
$\ddot{a}_{64:65} = 1 + a_{64:65} = 1 + 8.4129 = 9.4129$
$p_{64} = 1 - q_{64} = 1 - (8.685/1{,}000) = .991315$

Using the formula for consecutive life annuities due,

$$\ddot{a}_{64} = 1 + vp_{64}\,\ddot{a}_{65} = 1 + (1/1.07)(.991315)(10.8207) = 11.0250$$

Substituting into the present value from above,

$$PV = (6{,}000 \times 11.0250) + (6{,}000 \times 10.8207) - (2{,}000 \times 9.4129) = 112{,}248$$

Answer is C.

Question 20

The term cost for the 2004 withdrawal benefit is calculated as follows:

$93,084 = ($1,000 × v × $q_{40}^{(w)}$) × 1,000 participants $\Rightarrow$ $q_{40}^{(w)} = .09960$

The present value of the bonus payment as of 1/1/2004 is calculated as follows:

$2,167,971 = ($3,000 × v^2 × ${}_2p_{40}^{(T)}$) × 1,000 participants $\Rightarrow$ ${}_2p_{40}^{(T)} = .82737$

The value of $q_{40}^{(d)}$ must be determined.

$${}_2p_{40}^{(T)} = p_{40}^{(T)} \times p_{41}^{(T)} = (1 - q_{40}^{(w)} - q_{40}^{(d)}) \times (1 - q_{41}^{(w)} - q_{41}^{(d)})$$

Substituting,

$$.82737 = (1 - .09960 - q_{40}^{(d)}) \times (1 - 200\,q_{40}^{(d)} - .0007) = (.9004 - q_{40}^{(d)}) \times (.9993 - 200\,q_{40}^{(d)})$$

$$.82737 = 200\left(q_{40}^{(d)}\right)^2 - 181.0793\,q_{40}^{(d)} + .89977$$

$$200\left(q_{40}^{(d)}\right)^2 - 181.0793\,q_{40}^{(d)} + .0724 = 0$$

Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, a = 200, b = -181.0793, and c = .0724.

Substituting,

$$q_{40}^{(d)} = \frac{181.0793 \pm \sqrt{(-181.0793)^2 - (4)(200)(.0724)}}{(2)(200)} = .9050 \text{ or } .0004$$

Clearly the result of .9050 makes no sense, since that would imply that over 90% of the active participants die between age 40 and 41, impossible since ${}_2p_{40}^{(T)} = .82737$. Therefore, $q_{40}^{(d)} = .0004$

The term cost for the 2004 death benefit is calculated as follows:

($2,000 × v × $q_{40}^{(d)}$) × 1,000 participants = 748

Answer is A.

Question 21

${}_1E_{76} = vp_{76} = 0.9 \quad \Rightarrow \quad p_{76} = 0.927$ and $q_{76} = 0.073$

$A_{76} = vq_{76} + v^2\ {}_{1|}q_{76} + v^3\ {}_{2|}q_{76} + \ldots = vq_{76} + vp_{76}A_{77}$

Substituting,

$0.8 = (0.073/1.03) + (0.927/1.03)A_{77} \quad \Rightarrow \quad A_{77} = .81014$

Answer is C.

Question 22

When the force of mortality doubles, the probability of survival increases exponentially by a factor of 2.

$e_{107} = p_{107} + {}_2p_{107} = p_{107} \times (1 + p_{108}) = p_{107} \times (1 + e_{108})$

Substituting,

$0.6 = p_{107} \times (1 + 0.2) \quad \Rightarrow \quad p_{107} = .5$

Since life expectancy at age 109 is 0, there are no lives left at age 109. Therefore,

$p_{108} = e_{108} = .2$

Under the new table with double the force of mortality,

$Y = (p_{107})^2 + (p_{107} \times p_{108})^2 = (.5)^2 + (.5 \times .2)^2 = .26$

Answer is B.

Question 23

The purchase price of a bond is equal to the present value of the maturity (redemption) value plus the present value of the coupons. The redemption amount is equal to 5 equal annual installments of \$200 (one-fifth of the face amount) beginning at the end of the 10^{th} year. The coupons of the given bond are paid semi-annually in the amount of \$30 every six months (6% of \$1,000 divided by two).

Since the yield rate is an annual rate, the equivalent semi-annual rate must be determined.

For $i = .05$, $i^{(2)}/2 = 1.05^{1/2} - 1 = .024695$

Since the coupons are paid semiannually, the equivalent annual coupon payment as of the beginning of the year could be determined as the present value of the two coupon payments.

$$\text{Annual coupon payment on January 1 each year} = \text{Semi-annual coupon} \times a_{\overline{2}|.024695}$$
$$= \text{Semi-annual coupon} \times 1.9283$$

Using the annual yield rate of 5%, the present value of the redemption amount is:

$$\text{PV redemption} = 200\, a_{\overline{5}|.05}\, v^9 = 200 \times 4.3295 \times .6446 = 558$$

For each of the first 10 years, the coupons are paid on the entire face amount of the bond. However, beginning with the 11^{th} year, the coupons reduce as the face amount of the bond reduces by \$200 per year of redemption. So, the coupons in the 11^{th} year are \$24 every six months. The coupons in the 12^{th} year are \$18 every six months. And so on.

Using the annual yield rate of 5%, the present value of the coupons is:

$$\text{PV coupons} = (30 \times 1.9283 \times \ddot{a}_{\overline{10}|.05}) + (6 \times 1.9283 \times v^{10} \times (D\ddot{a})_{\overline{4}|.05})$$
$$= 469 + 68$$
$$= 537$$

Note that $(D\ddot{a})_{\overline{4}|.05} = \dfrac{4 - a_{\overline{4}|}}{.05} \times 1.05.$

The purchase price of the bond is:

Price = PV redemption + PV coupons = 558 + 537 = 1,095

Answer is C.

Question 24

Recall that the formula for whole life insurance at consecutive ages is:

$A_x = vq_x + vp_x A_{x+1}$

Substituting into the equality $A_{x+1} = vq_{x+1} + vp_{x+1}A_{x+2}$,

$.19 = (.01125)v + (.98875)(.20)v \quad \Rightarrow \quad v = .909091$

Substituting into the equality $A_x = vq_x + vp_x A_{x+1}$,

$.18 = (.909091)q_x + (.909091)(1 - q_x)(.19) \quad \Rightarrow \quad q_x = .009877$
$\Rightarrow \quad 1000\, q_x = 9.877$

Answer is A.

Question 25

Setting the present value of the two annuities equal to each other:

$$(Da)_{\overline{10}|} = (Ia)_{\overline{10}|} + 11\, a_{\overline{\infty}|}\, v^{10} \quad \Rightarrow \quad \frac{10 - a_{\overline{10}|}}{i} = \frac{\ddot{a}_{\overline{10}|} - 10v^{10}}{i} + \frac{11v^{10}}{i}$$

$$\Rightarrow \quad 10 - a_{\overline{10}|} = \ddot{a}_{\overline{10}|} - 10v^{10} + 11v^{10}$$

$$\Rightarrow \quad 10 - a_{\overline{10}|} = \ddot{a}_{\overline{10}|} + v^{10} = 1 + a_{\overline{10}|}$$

$$\Rightarrow \quad 2\, a_{\overline{10}|} = 9$$

$$\Rightarrow \quad a_{\overline{10}|} = 4.5$$

Solving for the interest rate, i = .18

Note that the interest rate can be determined using the financial functions of your calculator, or using interpolation.

Calculating the present value of the decreasing annuity,

$$(Da)_{\overline{10}|} = \frac{10 - a_{\overline{10}|}}{i} = 30.56$$

Answer is D.

Question 26

The product limit estimate is equal to the probability of survival, ignoring the actual time passed. In addition, terminations are ignored since the product limit estimate is measuring only deaths. New entrants are taken into account. Note that the concept of ignoring the actual passage of time here is similar to the concept used in determining the time-weighted rate of return.

The following are the important event points in time (points in time where there is either a death or a new entrant): Time 1.1, 2.3, 3.0, 3.2, and 6.0

The probability of survival to time 1.1 is 19/20 (since one of the original 20 entrants dies at that point).

That leaves 19. However, 9 terminate at time 1.5 (before the next death), so we find that at time 2.3, 1 of the remaining 10 dies. The probability of surviving this time period is 9/10. Note that the terminated group is completely removed from the computation.

That leaves 9. However, there is 1 new entrant at time 3.0 (before the next death), so we find that at time 3.2, 1 of the remaining 10 dies. The probability of surviving this time period is 9/10.

That leaves 9. However, 1 terminates at time 4.7 (before the next death), so we find that at time 6.0, 2 of the remaining 8 dies. The probability of surviving this time period is 6/8. Note that once again the terminated group is completely removed from the computation.

The product limit estimate of S(6) can now be determined as the product of the four probabilities of survival.

$Y = \frac{19}{20} \times \frac{9}{10} \times \frac{9}{10} \times \frac{6}{8} = .577125$

Answer is D.

See Section 7.6 of Survival Models and Their Estimation by Dick London for a discussion of Product Limit estimators.

Question 27

The monthly effective interest rate is $i^{(12)}/12 = .075/12 = .00625$

The interest in a particular payment of a loan can be determined using the following formula:

Interest = Payment × $(1 - v^n)$, where n = remaining loan payments including current payment.

Including the 54th payment, there are 67 payments remaining. The interest portion of the 54th payment is:

$100 = \text{Payment} \times (1 - v^{67}) = \text{Payment} \times (1 - .6587) \quad \Rightarrow \quad \text{Payment} = 293$

The outstanding balance of the loan is equal to the present value of the remaining payments. After the 90th payment, there are 30 payments left.

$P = 293\, a_{\overline{30}|.00625} = 7{,}992$

Answer is A.

Question 28

The purchase price of a bond is equal to the present value of the maturity (redemption) value plus the present value of the coupons. Let the amount of each semiannual coupon be represented by C.

The given yield rate must be converted to semiannual interest rates:

For i = .04, $i^{(2)}/2 = 1.04^{1/2} - 1 = .019804$

For $i = .05$, $i^{(2)}/2 = 1.05^{1/2} - 1 = .024695$

The purchase price of the bond at issue can be represented as follows using each yield rate:

$$\text{At 4\%, } P = 1{,}100\, v_{.04}^{10} + C\, a_{\overline{20}|.019804} \quad \Rightarrow \quad P = 743.12 + 16.3824C$$

$$\text{At 5\%, } P - 95.50 = 1{,}100\, v_{.05}^{10} + C\, a_{\overline{20}|.024695} \quad \Rightarrow \quad P = 770.82 + 15.6342C$$

$$743.12 + 16.3824C = 770.82 + 15.6342C \quad \Rightarrow \quad C = 37$$

$$r\% = \tfrac{37}{1{,}000} \times 2 = .074, \text{ or } 7.4\%$$

Answer is B.

Question 29

Under the uniform distribution of death assumption,

$$\mu_{x+s} = \frac{q_x}{1 - s \cdot q_x} \quad \Rightarrow \quad .5980 = \frac{q_{98}}{1 - .55 \cdot q_{98}}$$

$$\Rightarrow \quad .598 \times (1 - .55q_{98}) = q_{98}$$

$$\Rightarrow \quad q_{98} = .45$$

Under the uniform distribution of death assumption,

$$_yq_{x+s} = \frac{y \cdot q_x}{1 - s \cdot q_x} \quad \Rightarrow \quad .2486 = \frac{y \cdot q_{98}}{1 - .35 \cdot q_{98}}$$

$$\Rightarrow \quad .2486 \times (1 - .35q_{98}) = yq_{98}$$

$$\Rightarrow \quad .2486 \times (1 - (.35)(.45)) = .45y$$

$$\Rightarrow \quad y = .4654$$

Under a constant force of mortality, the force of mortality is the same over the age interval [98, 99].

$$Z = {}_yq_{x+s} = {}_{.4654}q_{98+s} = {}_{.4654}q_{98.5346} = 1 - (1 - q_{98})^{.4654} = 1 - (1 - .45)^{.4654} = .2429$$

Answer is A.

Note: The above formulas can be found on page 66 of "Survival Models and Their Estimation" by Dick London.

Question 30

The total lifetime of the lives currently age x in a stationary population is:

$xl_x + T_x$

The number of lives that die each year in a stationary population between the ages of 24 and 55 is $l_{24} - l_{55}$.

The total lifetime of those who die between age 24 and 55 is:

$(24l_{24} + T_{24}) - (55l_{55} + T_{55})$

In the given population, $l_{24} = 510$ since 510 lives are hired each year at age 24.

In the given population, $T_{24} - T_{55} = 15{,}000$.

The average age at death is equal to the ratio of the total lifetime to the number who die.

$$50 = \frac{(24l_{24} + T_{24}) - (55l_{55} + T_{55})}{l_{24} - l_{55}} = \frac{(24l_{24} - 55l_{55}) + (T_{24} - T_{55})}{l_{24} - l_{55}}$$

$$50 = \frac{(24 \times 510) - 55l_{55} + 15{,}000}{510 - l_{55}} \qquad \Rightarrow \qquad l_{55} = 348$$

Answer is A.

Note that this can also be solved without the use of stationary population notation. Since the population is stationary and the average age at death is 50, assume that all deaths occur each year at age 50. There are 510 new hires each year at age 24. Therefore, in any year, there are 510 people at each age from 24 through 49. At age 50, the deaths occur. Assume that there are D deaths each year. The number of people at age 50 (after the deaths occur) is 510 – D. That is also the number of people at each age from 51 through 54.

The total population of 15,000 can be written as the sum of the number of people at each age from 24 through 54. There are 26 ages from 24 through 49, and 5 ages from 50 through 54.

$$15{,}000 = (510 \times 26) + ((510 - D) \times 5) \qquad \Rightarrow \qquad D = 162$$

Since there are 162 deaths each year, the total number of employees at each age 50 and older is 348 (510 – 162). This is the number of retirees each year.

Question 31

$l_{26}^{(T)} = l_{25}^{(T)} \times (1 - q_{25}^{(1)} - q_{25}^{(2)}) \quad \Rightarrow \quad 6{,}500 = l_{25}^{(T)} \times (1 - .05 - .30) \quad \Rightarrow \quad l_{25}^{(T)} = 10{,}000$

$d_{26}^{(1)} = l_{26}^{(T)} \times q_{26}^{(1)} = 6{,}500 \times .05 = 325$

If $q_{25}^{(2)}$ is changed to 0.25,

$l_{26}^{(T)} = l_{25}^{(T)} \times (1 - q_{25}^{(1)} - q_{25}^{(2)}) = 10{,}000 \times (1 - .05 - .25) = 7{,}000$

$d_{26}^{(1)} = l_{26}^{(T)} \times q_{26}^{(1)} = 7{,}000 \times .05 = 350$

$Z = 350 - 325 = 25$

Answer is E.

Question 32

The purchase price of a bond is equal to the present value of the maturity (redemption) value plus the present value of the coupons. Let $i^{(2)}/2$ be the effective semiannual yield rate. Bond A has semiannual coupons of \$3, and Bond B has semiannual coupons of \$2.50.

Setting the purchase price of the bonds equal to each other,

$$100v^{40} + 3\,a_{\overline{40|}} = 125v^{40} + 2.5\,a_{\overline{40|}}$$
$$25v^{40} = 0.5\,a_{\overline{40|}} \quad \Rightarrow \quad s_{\overline{40|}} = 50$$

For $i^{(2)}/2 = 1.10\%$, $s_{\overline{40|}} = 49.9074$

For $i^{(2)}/2 = 1.11\%$, $s_{\overline{40|}} = 50.0110$

Interpolating,

$$i^{(2)}/2 = 1.10\% + \left(.01\% \times \frac{50 - 49.9074}{50.0110 - 49.9074}\right) = 1.1089\%$$

The yield rate is:

$i = 1.011089^2 - 1 = .0223$, or 2.23%

Answer is D.