Solutions to EA-1 Examination
Spring, 2003

Question 1

The interest rate credited for each of the 3 years should be converted to a rate of interest that matches the frequency of deposits for that year.

Recall that $i = \dfrac{d}{1-d}$

Year 1: $d^{(12)}/12 = .06/12 = .005$, and $\dfrac{i^{(12)}}{12} = \dfrac{d^{(12)}/12}{1-d^{(12)}/12} = \dfrac{.005}{1-.005} = .005025$

$i^{(2)}/2 = 1.005025^6 - 1 = .030532$

Year 2: $i^{(3)}/3 = .08/3 = .026667$, and $i = 1.026667^3 - 1 = .082153$

$i^{(4)}/4 = 1.082153^{1/4} - 1 = .019934$

Year 3: δ can be thought of as $i^{(m)}$ for a very large m. Let m be 10,000.

$i^{(10{,}000)}/10{,}000 = .07/10{,}000 = .000007$

$i = 1.000007^{10{,}000} - 1 = .072508$

$i^{(6)}/6 = 1.072508^{1/6} - 1 = .011735$

Account value after one year: $\$600\,\ddot{s}_{\overline{2}|.030532} = \$1{,}256$

Account value after two years: $(\$1{,}256 \times 1.019934^4) + \$300\,\ddot{s}_{\overline{4}|.019934} = \$2{,}620$

Account value after three years: $(\$2{,}620 \times 1.011735^6) + \$200\,\ddot{s}_{\overline{6}|.011735} = \$4{,}060$

Answer is D.

Question 2

$B = 10{,}000\,\bar{a}_{\overline{y|}}\, v^x$

$C = 10{,}000\,\bar{a}_{\overline{\infty|}}\, v^y v^x$

Since $B = C/2$, $10{,}000\,\bar{a}_{\overline{y|}}\, v^x = ½(10{,}000\,\bar{a}_{\overline{\infty|}}\, v^y v^x)$

$\bar{a}_{\overline{y|}} = ½(\bar{a}_{\overline{\infty|}}\, v^y)$

$\frac{1 - v^y}{\delta} = ½(\frac{1}{\delta} v^y)$

$1 - v^y = ½v^y$

$v^y = 2/3$, or .6667

Answer is D.

Question 3

The present value of the loan payments is equal to \$10,000. The equation of value is:

$10{,}000 = Pv + 1.03Pv^2 + 1.03^2Pv^3 + \ldots + 1.03^{29}Pv^{30}$

Multiplying each side of the equation by a factor of 1.03:

$$\begin{aligned}10{,}300 &= 1.03Pv + 1.03^2Pv^2 + \ldots + 1.03^{30}Pv^{30}\\ &= P(1.03v + 1.03^2v^2 + \ldots + 1.03^{30}v^{30})\\ &= P\,a_{\overline{30|}j} \qquad \text{(where } j = 1.075/1.03 - 1)\end{aligned}$$

$P = 622.62$

The outstanding balance of the loan after the 20th payment is equal to the present value of the future payments:

$$\begin{aligned}\text{Outstanding balance} &= 622.62 \times (1.03^{20}v + 1.03^{21}v^2 + \ldots + 1.03^{29}v^{10})\\ &= 622.62 \times 1.03^{19} \times (1.03v + 1.03^2v^2 + \ldots + 1.03^{10}v^{10})\\ &= 1{,}091.77 \times a_{\overline{10|}j} \qquad \text{(where } j = 1.075/1.03 - 1)\\ &= 8{,}695\end{aligned}$$

Principal repaid in first 20 payments = 10,000 – 8,695 = 1,305

$$\begin{aligned}\text{Total of first 20 payments} &= 622.62 \times (1 + 1.03 + 1.03^2 + \ldots + 1.03^{19})\\ &= 622.62\, s_{\overline{20|}.03}\\ &= 16{,}730\end{aligned}$$

The interest paid in the first 20 payments is equal to the total payments less the principal repaid.

X = Total of first 20 payments – outstanding balance = 16,730 – 1,305 = 15,425

Answer is C.

Question 4

$i^{(12)}/12 = 1.075^{1/12} - 1 = .006045$

The initial monthly mortgage payment is:

$300{,}000/a_{\overline{360}|.006045} = 2{,}047.34$

The outstanding balance after the 120th payment (including the additional $2,000 payment) is equal to the present value of the remaining payments less $2,000.

Outstanding balance = $2{,}047.34\,a_{\overline{240}|.006045} - 2{,}000 = 256{,}954$

$j^{(12)}/12 = 1.07^{1/12} - 1 = .005654$

The new monthly mortgage payment P is:

$P = 256{,}954/a_{\overline{240}|.005654} = 1{,}959.11$

Answer is C.

Question 5

The time-weighted rate of return for Smith is:

$$(1+i) = \frac{\text{Balance on } 6/30/2002^{*}}{\text{Balance on } 1/1/2002^{**}} \times \frac{\text{Balance on } 12/31/2002}{\text{Balance on } 7/1/2002^{**}}$$

$= (2{,}800{,}000/2{,}500{,}000) \times (4{,}180{,}000/3{,}800{,}000)$
$= 1.232$
$i = 23.2\%$

* Before transfer
** After contribution or transfer

The time-weighted rate of return for Jones is:

$$(1+j) = \frac{\text{Balance on } 6/30/2002^{*}}{\text{Balance on } 1/1/2002^{**}} \times \frac{\text{Balance on } 12/31/2002}{\text{Balance on } 7/1/2002^{**}}$$

$= (4{,}500{,}000/4{,}000{,}000) \times (3{,}500{,}000/3{,}500{,}000)$
$= 1.125$
$j = 12.5\%$

The dollar-weighted rate of return for the entire portfolio is:

Ending Balance = (Beginning Balance + Contribution)$(1 + k)$

$(4{,}180{,}000 + 3{,}500{,}000) = (2{,}500{,}000 + 2{,}500{,}000 + 1{,}500{,}000)\ (1 + k)$

$k = 18.15\%$

Note that the transfer stays within the entire portfolio, so it has no impact on the dollar-weighted rate of return.

X = ½ (23.2% + 12.5%) = 17.85%

Y = 18.15%

Z = Y – X = 18.15% - 17.85% = 0.3%

Answer is D.

Question 6

The fact that mortality follows the Balducci assumption produces the following formula for l_{x+t}, where $t < 1$:

$$l_{x+t} = [\, t \cdot \frac{1}{l_{x+1}} + (1-t) \cdot \frac{1}{l_x}\,]^{-1}$$ (see page 66 of “Survival Models and Their Estimation”)

As a result,

$$l_{60¾} = [\, \tfrac{3}{4} \cdot \frac{1}{l_{61}} + (1-\tfrac{3}{4}) \cdot \frac{1}{l_{60}}\,]^{-1} = 8{,}000$$

$$l_{61¾} = [\, \tfrac{3}{4} \cdot \frac{1}{l_{62}} + (1-\tfrac{3}{4}) \cdot \frac{1}{l_{61}}\,]^{-1} = 6{,}000$$

$X = l_{60¾} - l_{61¾} = 8{,}000 – 6{,}000 = 2{,}000$

Answer is B.

Question 7

T_x represents the number of members in a stationary population age x and older. In particular, T_0 represents the number of members in a stationary population age x and older (in other words, all members of the population). So, $T_0 = 9{,}800$.

The number of deaths before age 25 in any year is represented by $l_0 – l_{25}$. The number of deaths at age 25 and over in any year is represented by l_{25} (everyone who attains age 25 eventually dies at age 25 or over). Since there are four times as many deaths at age 25 and over as there are deaths at ages under 25,

$4 \times (l_0 – l_{25}) = l_{25} \quad \Rightarrow \quad 4l_0 = 5l_{25} \quad \Rightarrow \quad l_{25} = .8l_0$

Recall that the total lifetime for someone currently age x is: $xl_x + T_x$

The formula for the average age at death in a stationary population is the ratio of the total lifetime to the number who die. So, the average age at death for the lives now age x and over is:

$(xl_x + T_x)/ l_x$

The average age at death for those who are age 25 and over is 66.

$66 = (25l_{25} + T_{25})/ l_{25} \Rightarrow \quad 66l_{25} = 25l_{25} + T_{25}$

The average age at death for those who are under age 25 is 16.

$16 = (T_0 - (25l_{25} + T_{25}))/(l_0 - l_{25})$
$= (9{,}800 - 66l_{25}) / (l_0 - l_{25})$
$= (9{,}800 - 66(.8l_0) / (l_0 - .8l_0)$

$\Rightarrow \quad l_0 = 175 \qquad \Rightarrow \quad l_{25} = 140$

X = number who die each year under age 25 = $l_0 - l_{25} = 175 - 140 = 35$

Answer is E.

Question 8

For $i = .07$, $i^{(12)}/12 = 1.07^{1/12} - 1 = .005654$

Recall that $d = i/(1 + i)$.

So, $d^{(12)}/12 = (i^{(12)}/12)/(1 + (i^{(12)}/12)) = .005654/1.005654 = .005622$

And, $d^{(12)} = .005622 \times 12 = .067464$

$$\ddot{a}_{\overline{2}|}^{(12)} = \frac{1-v^2}{d^{(12)}} = 1.875982 \qquad \ddot{a}_{\overline{3}|}^{(12)} = \frac{1-v^3}{d^{(12)}} = 2.722965 \qquad \ddot{a}_{\overline{5}|}^{(12)} = \frac{1-v^5}{d^{(12)}} = 4.254326$$

The present value of annuities A and B are equal.

$$1{,}000 \times 12\,\ddot{a}_{60}^{(12)} = X \times 12(\ddot{a}_{\overline{5}|}^{(12)} + {}_{5|}\ddot{a}_{60}^{(12)}) \qquad \Rightarrow \qquad X = 993.60$$

The present value of annuities A and C are equal.

$$1{,}000 \times 12\,\ddot{a}_{60}^{(12)} = [(993.60 + 300) \times 12\,\ddot{a}_{\overline{3}|}^{(12)}] + [Y \times 12(\ddot{a}_{\overline{2}|}^{(12)} v^3 + {}_{5|}\ddot{a}_{60}^{(12)})]$$
$\Rightarrow \quad Y = 901.62$

Answer is D.

Question 9

For $i = .07$, $i^{(12)}/12 = 1.07^{1/12} - 1 = .005654$

Recall that $d = i/(1 + i)$.

So, $d^{(12)}/12 = (i^{(12)}/12)/(1 + (i^{(12)}/12)) = .005654/1.005654 = .005622$

And, $d^{(12)} = .005622 \times 12 = .067464$

$$\ddot{a}^{(12)}_{\overline{3}|} = \frac{1 - v^3}{d^{(12)}} = 2.722965$$

It is given that there are 1,000 active participants on 1/1/2003, and that $p_{60}^{(\text{total})} = 0.990$.

Therefore, the number of active participants on 1/1/2004 is: $1{,}000 \times p_{60}^{(\text{total})} = 990$

For participants who die during 2004, the death benefit is payable beginning on 1/1/2005, for 3 years. The amount of the death benefit (present value on 1/1/2005) is:

$1{,}000 \times 12\,\ddot{a}^{(12)}_{\overline{3}|} = 32{,}676$

$q_{61}^{(\text{total})} = 1 - p_{61}^{(\text{total})} = 1 - 0.985 = 0.015$

$q_{61}^{(\text{total})} = q_{61}^{(\text{mort})} + q_{61}^{(\text{dis})} \quad \Rightarrow \quad 0.015 = q_{61}^{(\text{mort})} + 0.011 \quad \Rightarrow \quad q_{61}^{(\text{mort})} = 0.004$

Recall the formula used to calculate single decrements from a two-decrement table:

$q_x'^{(1)} = q_x^{(1)}/(1 - ½\, q_x^{(2)})$

Note that the above formula assumes a uniform distribution of decrement. Using the formula,

$q_{61}'^{(\text{mort})} = q_{61}^{(\text{mort})}/(1 - ½\, q_{61}^{(\text{dis})}) = 0.004/(1 - ½\,(0.011)) = .004022$, and
$q_{61}'^{(\text{dis})} = q_{61}^{(\text{dis})}/(1 - ½\, q_{61}^{(\text{mort})}) = 0.011/(1 - ½\,(0.004)) = .011022$

X is the one-year term cost as of 1/1/2004 (present value of the death benefit for deaths that occur in 2004). Note that death must occur before disability occurs, and since it is assumed that disabilities occur uniformly throughout the year, it is assumed that half of the people who will become disabled in 2004 do so before they can die.

$X = 32{,}676\, v\, q_{61}'^{(\text{mort})}(1 - ½\, q_{61}'^{(\text{dis})}) \times 990$ active participants $= 121{,}927$

Answer is D.

Note that a shorter way to solve this question is to simply use the multiple decrement table to value the one-year term cost. Since the probability of death in the multiple decrement table at age 61 is 0.004, this means that $0.004 \times 990 = 3.96$ people are expected to die in 2004.

The one-year term cost is:

$X = 32{,}676v \times 3.96 \text{ deaths} = 121{,}932$

The numerical difference is due to rounding.

Question 10

The equation of value is:

$100{,}000 = X\ddot{s}_{\overline{14|}}\,1.07^{28} + (X - 100)\ddot{s}_{\overline{10|}} \qquad \Rightarrow \qquad X = 579.17$

Answer is C.

Question 11

The semi-annual rate of interest in the fund in which the interest is invested is:

$i^{(2)}/2 = 1.06^{1/2} - 1 = .029563$

Assume that the initial single deposit is \$1. The deposit into the reinvestment fund every six months is $i/2$. After 15 years, the amount in the original fund is \$1, and the amount in the reinvestment fund is $(i/2)\,s_{\overline{30|}.029563}$. The sum of the two funds must be equal to an amount such that the effective annual rate of return is 7.56%.

$1 + (i/2)\,s_{\overline{30|}.029563} = 1.0756^{15} \quad \Rightarrow \quad i = .084$, or 8.4%

Answer is C.

Question 12

Amortization using method I: $\qquad 10{,}000/a_{\overline{10|}.08} = 1{,}490$

Amortization using method II:

$[8\% \times 10{,}000] + 10{,}000/s_{\overline{10|}.06} = 800 + 759 = 1{,}559$

Amortization using method III:

$[8\% \times 10{,}000] + 10{,}000/s_{\overline{10|}.08} = 800 + 690 = 1{,}490$

Amortization using method IV:

$[8\% \times 10{,}000] + 10{,}000/s_{\overline{10|}.12} = 800 + 570 = 1{,}370$

$IV < I = III < II$

Answer is E.

Question 13

The principal portion of any level loan payment is equal to the product of the payment and v^n, where n is equal to the number of remaining payments (including the current payment being made). Since the loan in this question will have 20 level annual payments, the number of remaining payments starting with the 13th payment is 8, and the number of remaining payments starting with the 18th payment is 3. Therefore, if P represents the annual loan payment,

$Pv^8 = \$54.40$ and $Pv^3 = \$70.09$

Taking ratios,

$Pv^8 / Pv^3 = \$54.40/\$70.09 \quad \Rightarrow \quad v^5 = .776145$

Solving for the annual rate of interest, $i = .052$

Solving for the annual payment, $P = \$54.40/ v^8 = \$81.61$

Total payments made during life of loan = $\$81.61 \times 20 = \$1{,}632.20$

Amount of loan = $\$81.61 \times a_{\overline{20|}} = \$1{,}000.00$

The total interest paid over the life of the loan is equal to the difference between the total loan payments and the original amount of the loan.

Total interest paid = \$1,632.20 - \$1,000.00 = \$632.20

Answer is B.

Question 14

Using the information from item (iii):

$\bar{a}_{\overline{20|}} = 1.4\,\bar{a}_{\overline{10|}} \quad \Rightarrow \quad (1 - v^{20})/\delta = 1.4(1 - v^{10})/\delta$

$\Rightarrow \quad (1 - v^{20}) = 1.4(1 - v^{10})$

$\Rightarrow \quad v^{20} - 1.4v^{10} + .4 = 0$

Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, a = 1, b = -1.4, and c = .4.

Substituting,

$$v^{10} = \frac{1.4 \pm \sqrt{(-1.4)^2 - (4)(1)(.4)}}{(2)(1)} = 1.00 \text{ or } .4$$

Clearly the result of 1.00 makes no sense, since that would imply a 0% interest rate. Using the result of .4,

$v^{10} = .4 \quad \Rightarrow \quad i = .095958$

δ can be thought of as $i^{(m)}$ for a very large m. Let m be 10,000.

$\delta = [(1.095958)^{1/10{,}000} - 1] \times 10{,}000 = .091629$

Since item (ii) says that the forces of interest and mortality are the same, $\mu = .091629$.

Recall the formula $p_x = e^{-\mu_x}$. Substituting for μ_x,

$p_x = e^{-.091629} = .912444$

Since item (i) says that the force of mortality is constant at all ages, $p_x = .912444$ at all ages.

X is the probability that someone age 20 will live 20 years and then die within the next 10 years.

$X = {}_{20|10}q_{20} = {}_{20}p_{20} - {}_{30}p_{20} = .912444^{20} - .912444^{30} = .096$

Answer is D.

Question 15

The present value of annuity A is: $1{,}000v + 1{,}100v^3 + \ldots + 1{,}900v^{19}$
The present value of annuity B is: $2{,}000v + (.95)(2{,}000)v^3 + \ldots + (.95^9)(2{,}000)v^{19}$
The present value of annuity C is: $X\ddot{a}_{\overline{\infty}|} = X(1/d) = X(1/(i/(1+i)) = 15.2857X$

Let $j = 1.07^2 - 1 = .1449$, and $k = 1.07^2/.95 - 1 = .2052$

Recall that $(Ia)_{\overline{n}|} = \dfrac{\ddot{a}_{\overline{n}|} - nv^n}{i}$.

Setting the sum of the present values of annuities A and B equal to the present value of annuity C:

$$\begin{aligned}15.2857X &= (1{,}000v + 1{,}100v^3 + \ldots + 1{,}900v^{19}) \\ &\quad + (2{,}000v + (.95)(2{,}000)v^3 + \ldots + (.95^9)(2{,}000)v^{19}) \\ &= v(1{,}000 + 1{,}100v^2 + \ldots + 1{,}900v^{18}) + 2{,}000v(1 + (.95)v^2 + \ldots + (.95^9)v^{18}) \\ &= v(1{,}000\,\ddot{a}_{\overline{10}|j} + 100\,(Ia)_{\overline{9}|j}) + 2{,}000v\,\ddot{a}_{\overline{10}|k} \\ &= v(5{,}859 + 2{,}002) + 2{,}000v(4.9649) \\ &= 16{,}627\end{aligned}$$

$X = 1{,}088$

Answer is B.

Question 16

The purchase price of a bond is equal to the present value of the maturity value plus the present value of the coupons. The bond in this question matures after one year for \$1,000, and pays a \$100 coupon (10% coupon rate) at the end of the year. Payment is only made if the bond does not go into default, so the present values must take into account the probability of default (or non-default).

$\$980 = \$1{,}000v\text{p} + \$100v\text{p} \quad \Rightarrow \quad \text{p} = 980/(1{,}100v) = .9533$
$\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \quad \text{q} = 1 - \text{p} = .0467$

Answer is E.

Question 17

This annuity provides \$10,000 if both the participant and the annuitant are alive, and \$5,000 if only one is alive. This can be written as saying that the annuity provides \$5,000 if at least one of the participant and the annuitant are alive, and \$5,000 if both are alive.

The present value of this annuity is:

$$
\begin{aligned}
5{,}000\, a_{\overline{62:65}:\overline{2}|} + 5{,}000\, a_{62:65:\overline{2}|} &= 5{,}000[(a_{62:\overline{2}|} + a_{65:\overline{2}|} - a_{62:65:\overline{2}|}) + a_{62:65:\overline{2}|}] \\
&= 5{,}000(a_{62:\overline{2}|} + a_{65:\overline{2}|}) \\
&= 5{,}000(vp_{62} + v^2\, {}_2p_{62} + vp_{65} + v^2\, {}_2p_{65}) \\
&= 17{,}307
\end{aligned}
$$

Answer is B.

Question 18

Recall the approximation (for a triple decrement table): $p_x^{(T)} = p_x'^{(1)} \times p_x'^{(2)} \times p_x'^{(3)}$

$$p_{54}^{(T)} = p_{54}'^{(w)} \times p_{54}'^{(d)} \times p_{54}'^{(r)} = .96 \times .9956 \times 1.0 = .955776$$

$$l_{55}^{(T)} = l_{54}^{(T)} \times p_{54}^{(T)} = 1{,}000 \times .955776 = 955.776$$

Since withdrawals and deaths are both uniformly distributed between consecutive integral ages, the following equality holds (this is the case from formula 10.6.3 in the "Actuarial Mathematics" text, where there are two decrements):

$$q_x^{(d)} = q_x'^{(d)} \times (1 - ½\, q_x'^{(w)})$$

Note that retirements can be ignored since they occur at the beginning of the year, and the lives that retire are not subject to either death or withdrawal within the participant group for that year.

So, $q_{54}^{(d)} = q_{54}'^{(d)} \times (1 - ½\, q_{54}'^{(w)}) = .0044 \times (1 - ½(.04)) = .004312$

$$d_{54}^{(d)} = l_{54}^{(T)} \times q_{54}^{(d)} = 1{,}000 \times .004312 = 4.312$$

There are no withdrawals in 2004 (at age 55), so the probability of death within the multiple decrement table is equal to the absolute rate of death.

Since 15% of the remaining population retires on 1/1/2004 (at the beginning of the year), the actual number of participants still subject to death or withdrawal during 2004 is:

$l_{55}^{(T)} \times 85\% = 955.776 \times .85 = 812.410$

The expected number of deaths in 2004 (at age 55) is:

$d_{55}^{(d)} = 812.410 \times q_{55}^{(d)} = 812.410 \times .0049 = 3.981$

The total expected number of deaths during 2003 and 2004 is:

$4.312 + 3.981 = 8.293$

Answer is C.

Question 19

The present value of the two actuarially equivalent forms of benefit can be set equal to each other.

$$10{,}000\,\ddot{a}_{65} = Y\,\ddot{a}_{62:65} + .75Y\,\ddot{a}_{62|65} + .75Y\,\ddot{a}_{65|62}$$

$$\begin{aligned} 10{,}000 \times 10.0426 &= 8.7060Y + .75Y(\ddot{a}_{65} - \ddot{a}_{62:65}) + .75Y(\ddot{a}_{62} - \ddot{a}_{62:65}) \\ &= 8.7060Y + 1.00245Y + 1.49355Y \end{aligned}$$

$Y = 8{,}965$

Answer is D.

Question 20

The principal portion of any level loan payment is equal to the product of the payment and v^n, where n is equal to the number of remaining payments (including the current payment being made). Similarly, the interest portion of any level loan payment is equal to the product of the payment and $(1 - v^n)$, where n is equal to the number of remaining payments (including the current payment being made).

So, for example, the interest portion of the first payment of the loan in this question is equal to $(1{,}000 \times (1 - v^{25}))$, since there are 25 payments remaining, including the first payment.

$$
\begin{aligned}
X &= 1{,}000 \times [\, v_{.12}(1 - v_{.08}^{25}) + v_{.12}^{2}(1 - v_{.08}^{24}) + \ldots + v_{.12}^{25}(1 - v_{.08})] \\
&= 1{,}000 \times [(v_{.12} + v_{.12}^{2} + \ldots + v_{.12}^{25}) - (v_{.12}\, v_{.08}^{25} + v_{.12}^{2}\, v_{.08}^{24} + \ldots + v_{.12}^{25}\, v_{.08})] \\
&= 1{,}000 \times [\, a_{\overline{25}|.12} - v_{.08}^{26}\left(\frac{v_{.12}}{v_{.08}} + \frac{v_{.12}^{2}}{v_{.08}^{2}} + \ldots + \frac{v_{.12}^{25}}{v_{.08}^{25}}\right)] \\
&= 1{,}000 \times [\, a_{\overline{25}|.12} - v_{.08}^{26}(v_{i'} + v_{i'}^{2} + \ldots + v_{i'}^{25})] \qquad \text{(where } i' = 1.12/1.08 - 1 = .037037\text{)} \\
&= 1{,}000 \times [\, a_{\overline{25}|.12} - v_{.08}^{26}\, a_{\overline{25}|.037037}] \\
&= 1{,}000 \times [7.843139 - (.135202 \times 16.123058)] \\
&= 5{,}663
\end{aligned}
$$

Answer is C.

Question 21

The purchase price of a bond is equal to the present value of the maturity value plus the present value of the coupons. Since coupons are paid semi-annually in the amount of $40 every six months (8% of $1,000 divided by two), the yield rate must be converted to a semi-annual interest rate.

$i^{(2)}/2 = 1.09^{1/2} - 1 = .044031$

Purchase price of the bond = $1{,}000v^{30} + 40\, a_{\overline{30}|.044031} = 275 + 659 = 934$

The yield rate for the buyer must be converted to a semi-annual interest rate.

$j^{(2)}/2 = 1.0825^{1/2} - 1 = .040433$

Purchase price of the bond = $1{,}000v^{20} + 40\, a_{\overline{20}|.040433} = 453 + 542 = 995$

Smith receives $995 after 5 years, and received 10 semiannual coupon payments of $40 each. The equation illustrating Smith's yield (at a semi-annual effective interest rate $X^{(2)}/2$) is:

$934 = 995v^{10} + 40\, a_{\overline{10}|}$

If X = 9.5%, then $X^{(2)}/2 = 1.095^{1/2} - 1 = .046422$, and the present value of Smith's payments is:

$995v^{10} + 40\, a_{\overline{10}|.046422} = 632 + 314 = 946$

If X = 9.9%, then $X^{(2)}/2 = 1.099^{1/2} - 1 = .048332$, and the present value of Smith's payments is:

$995v^{10} + 40\, a_{\overline{10}|.048332} = 621 + 311 = 932$

Clearly, X must be greater than 9.5% and less than 9.9%, which is range D.

A more exact answer can be found using interpolation.

$$X^{(2)}/2 = .046422 + (.048332 - .046422) \times \left(\frac{946 - 934}{946 - 932}\right) = .048059$$

$X = 1.048059^2 - 1 = .098428$

Answer is D.

Question 22

For a constant force of decrement, μ_x = -ln p_x. See page 66 of "Survival Models and Their Estimation".

$$\mu_x^{(w)} = -\ln p_x'^{(w)} = .2 \;\Rightarrow\; \ln p_x'^{(w)} = -.2 \;\Rightarrow\; e^{\ln p_x'^{(w)}} = e^{-.2}$$

$$\Rightarrow\; p_x'^{(w)} = .818731 \;\Rightarrow\; q_x'^{(w)} = .181269$$

Using the standard formula to convert from single decrements to multiple decrements,

$$q_x^{(w)} = q_x'^{(w)} \left(1 - ½\, q_x'^{(d)}\right) = .181269(1 - ½\,(.03)) = .178550$$

Note that this formula applies since deaths are uniformly distributed between each integral age (allowing the use of the ½ adjustment).

Using another standard formula,

$$p_x^{(T)} = p_x'^{(w)} \times p_x'^{(d)} = .818731 \times (1 - .03) = .794169$$

$$q_x^{(w)} + q_x^{(d)} = q_x^{(T)} \;\Rightarrow\; q_x^{(d)} = (1 - .794169) - .178550 = .027281$$

Answer is B.

Note that there is actually a flaw in the above solution. It is given that withdrawals are subject to a constant force of mortality, rather than the usual uniform distribution. Therefore, the standard approximation formula to convert from single decrements to multiple decrements given above is really not accurate. The exact solution to this question can only be obtained by the use of first principles to obtain the multiple decrement value $q_x^{(d)}$. This involves the use of integration, which has never been required to solve an exam question in the past. The following is the correct solution to this question.

The general formula for a specific probability of decrement in a multiple decrement table (Formula 14.24 in the Jordan text is:

$$q_x^{(1)} = \int_0^1 {}_t p_x^{(T)} \mu_{x+t}^{(1)} dt$$

Since $p_x^{(T)} = p_x'^{(w)} \times p_x'^{(d)}$ and ${}_t p_x'^{(d)} \mu_{x+t}^{(d)} = q_x'^{(d)}$ (since mortality **is** subject to a uniform distribution – note that this formula is from the table on page 66 of the London text), we have:

$$q_x^{(d)} = \int_0^1 {}_t p_x^{(T)} \mu_{x+t}^{(1)} dt = \int_0^1 {}_t p_x'^{(d)} \mu_{x+t}^{(d)} {}_t p_x'^{(w)} dt = q_x'^{(d)} \int_0^1 {}_t p_x'^{(w)} dt$$

It is given in the table on page 66 of the London text for an assumed constant force of mortality that ${}_t p_x'^{(w)} = e^{-\mu t}$. Since it is given in the question that $\mu_x^{(w)} = .2$, then ${}_t p_x'^{(w)} = e^{-.2t}$.

Summarizing and integrating,

$$\begin{aligned} q_x^{(d)} &= q_x'^{(d)} \int_0^1 {}_t p_x'^{(w)} dt \\ &= q_x'^{(d)} \int_0^1 e^{-.2t} dt \\ &= q_x'^{(d)} [\frac{e^{-.2t}}{-.2} \text{(evaluated from 0 to 1)}] \\ &= q_x'^{(d)} \frac{e^{-.2} - e^0}{-.2} \\ &= q_x'^{(d)} \frac{.818731 - 1}{-.2} \end{aligned}$$

= (.03)(.906345)
= .02719

This is also within the range of choice B, and is the more correct solution.

Question 23

Recall the following from first principles:

$\ddot{a}_{x:\overline{n}|} = 1 + p_x v + {}_2p_x v^2 + \ldots + {}_{n-1}p_x v^{n-1}$

$(DA)_{x:\overline{n}|} = n q_x v + (n-1)\ {}_{1|}q_x v^2 + \ldots + {}_{n-1|}q_x v^n$

Since the annuity payments are made on the first day of the year, the death benefit under the "refund" annuity option is 11Z if death occurs in the first year, 10Z if death occurs in the second year, and so on for a total of 11 years.

Since the two forms of payment are actuarially equivalent, their present values can be set equal.

$100 \times (\ddot{a}_{\overline{10|}} + {}_{10|}\ddot{a}_x) = Z\ddot{a}_x + Z((DA)_{x:\overline{11|}})$

$100 \times (\ddot{a}_{\overline{10|}} + \ddot{a}_x - \ddot{a}_{x:\overline{10|}}) = Z\ddot{a}_x + Z(11q_xv + 10\ {}_{1|}q_xv^2 + \ldots + {}_{10|}q_xv^{11})$

$100 \times (\ddot{a}_{\overline{10|}} + \ddot{a}_x - (1 + p_xv + {}_2p_xv^2 + \ldots + {}_9p_xv^9))$

$= Z\ddot{a}_x + Z(11q_xv + 10\ {}_{1|}q_xv^2 + \ldots + {}_{10|}q_xv^{11})$

$100 \times (7.5152 + 9.3 - (1 + .98v + (.98v)^2 + \ldots + (.98v)^9))$

$= 9.3Z + Z(11(.02v) + 10(.98)(.02)v^2 + \ldots + (.98^{10})(.02)v^{11})$

$100 \times (16.8152 - (1 + .98v + (.98v)^2 + \ldots + (.98v)^9))$

$= 9.3Z + .02Z(11v + 10(.98)v^2 + \ldots + (.98^{10})v^{11})$

$100 \times (16.8152 - (1 + .98v + (.98v)^2 + \ldots + (.98v)^9))$

$= 9.3Z + .02Z(11v + 10(.98)v^2 + \ldots + (.98^{10})v^{11})$

$100 \times (16.8152 - \ddot{a}_{\overline{10|}i}) = 9.3Z + (.02/.98)Z((Da)_{\overline{11|}i})$ (where i = 1.07/.98 – 1 = .091837)

$100 \times (16.8152 - 6.9507) = 9.3Z + (.02/.98)Z(\frac{11 - a_{\overline{11|}}}{.091837})$

Z = 96.28

Answer is D.

Question 24

Using the standard formula to convert from single decrements to multiple decrements,

$q_{61}^{(2)} = q_{61}^{\prime(2)}\ (1 - ½\ q_{61}^{\prime(1)}) = .1(1 - ½\,(.2)) = .09$

Some of the blank items in the given table can be calculated.

$l_{60}^{(T)} \times p_{60}^{(T)} = l_{61}^{(T)} \quad \Rightarrow \quad l_{60}^{(T)} = 1{,}725/.92 = 1{,}875$

$l_{62}^{(T)} = l_{61}^{(T)} \times p_{61}^{\prime(1)} \times p_{61}^{\prime(2)} \quad \Rightarrow \quad l_{62}^{(T)} = 1{,}725 \times .8 \times .9 = 1{,}242$

$l_{60}^{(T)} - d_{60}^{(1)} - d_{60}^{(2)} = l_{61}^{(T)} \quad \Rightarrow \quad d_{60}^{(2)} = 1{,}875 - 120 - 1{,}725 = 30$

$d_{61}^{(2)} = l_{61}^{(T)} \times q_{61}^{(2)} = 1{,}725 \times .09 = 155$

${}_3q_{60}^{(2)} = (d_{60}^{(2)} + d_{61}^{(2)} + d_{62}^{(2)})/l_{60}^{(T)} = (30 + 155 + 77)/1{,}875 = .1397$

Answer is D.

Question 25

When there is a constant force of mortality, ${}_{.5}q_x = 1 - (p_x)^{.5}$, and ${}_{.5}q_{x+.5} = 1 - (p_x)^{.5}$. (See page 66 of "Survival Models and Their Estimation".)

In this question, ${}_{.5}q'^{(d)}_{65} = 1 - .96^{.5} = .020204$, and ${}_{.5}q'^{(d)}_{65.5} = 1 - .96^{.5} = .020204$.

So, the probability of death for Smith during the first half of 2003 is equal to .020204, and the probability of death for Smith during the last half of 2003 is equal to .979796 × .020204 = .019796 (the probability of surviving the first half of the year multiplied by the probability of dying during the last half of the year). Since all retirements occur in the middle of the year, there is a 50% chance of retirement at age 65.5 for Smith. The probability of death during 2003 is equal to the probability of dying during the first half of the year plus the probability of dying during the last half of the year (if retirement does not occur at age 65.5). This is represented by .020204 + (.5)(.019796) = .030102.

The present value of the death benefit for Smith is:

$10{,}000v \times q^{(d)}_{65} = 10{,}000 \times 1/1.07 \times .030102 = 281.33$

Answer is A.

Question 26

The present value of this annuity is:

$$
\begin{aligned}
\ddot{a}_{\overline{XY:\overline{20|}}} + {}_{20|}\ddot{a}_{XY} &= \ddot{a}_{X:\overline{20|}} + \ddot{a}_{Y:\overline{20|}} - \ddot{a}_{XY:\overline{20|}} + {}_{20|}\ddot{a}_{XY} \\
&= (\ddot{a}_X - {}_{20|}\ddot{a}_X) + (\ddot{a}_Y - {}_{20|}\ddot{a}_Y) - (\ddot{a}_{XY} - {}_{20|}\ddot{a}_{XY}) + {}_{20|}\ddot{a}_{XY} \\
&= \ddot{a}_X + \ddot{a}_Y - \ddot{a}_{XY} - {}_{20|}\ddot{a}_X - {}_{20|}\ddot{a}_Y + (2 \times {}_{20|}\ddot{a}_{XY}) \\
&= 10.5\text{, or } \$10.50
\end{aligned}
$$

Answer is D.

Question 27

From the given statement,

$$v^{5/12} = 99/100 \qquad \Rightarrow \qquad \left(\frac{1}{1+i}\right)^{\frac{5}{12}} = 99/100$$

Recall that $d = i/(1 + i)$ and that $i^{(m)}/m = (1 + i)^{1/m} - 1$

$$\text{So, } d^{(4)} = 4 \times (d^{(4)}/4) = 4 \times \left(\frac{i^{(4)}/4}{1 + i^{(4)}/4}\right) = 4 \times \left(\frac{(1+i)^{1/4} - 1}{1 + (1+i)^{1/4} - 1}\right) = 4 \times \left(\frac{(1+i)^{1/4} - 1}{(1+i)^{1/4}}\right)$$

$$= 4 \times \left(1 - \frac{1}{(1+i)^{1/4}}\right) = 4 \times \left(1 - \frac{1}{(1+i)^{5/12 \cdot 3/5}}\right) = 4 \times \left(1 - \frac{1}{(99/100)^{3/5}}\right)$$

Answer is A.

Question 28

$\ddot{a}_{\overline{10|}} = [(1 - v^{10})/.05] \times (1.05) = 8.107822$

The present value of the two alternative forms of benefit can be set equal to each other.

$5{,}000 v^{20}\ {}_{20}p_{45}\ (\ddot{a}_{\overline{10|}} + {}_{10|}\ddot{a}_{65}) = Y v^{10}\ {}_{10}p_{45}\ \ddot{a}_{55}$

$5{,}000 v^{20}\ ({}_{10}p_{45})({}_{10}p_{55})(\ddot{a}_{\overline{10|}} + v^{10}\ {}_{10}p_{65}\,\ddot{a}_{75}) = Y v^{10}\ {}_{10}p_{45}\ \ddot{a}_{55}$

Substituting the given values, Y = 2,764

Answer is B.