Solutions to EA-1 Examination
Spring, 2002

Question 1

$d^{(4)}$ = .076225, and $\dfrac{d^{(4)}}{4}$ = .019056

Recall that $i = \dfrac{d}{1-d}$

$$i^{(4)}/4 = \frac{\frac{d^{(4)}}{4}}{1-\frac{d^{(4)}}{4}} = \frac{.019056}{1-.019056} = .019426$$

$i^{(2)}/2 = 1.019426^2 - 1 = .039229$ and $i = 1.039229^2 - 1 = .08$

A = \$75,000/ $\ddot{a}_{\overline{5}|.08}$ = \$17,393

B = (\$75,000/ $a_{\overline{10}|.039229}$) × 2 = \$18,422

B – A = \$18,422 - \$17,393 = \$1,029

Answer is C.

Question 2

The accumulated value of the fund after 10 years is:

$$10{,}000\, s_{\overline{10}|} + 500\,(Is)_{\overline{9}|} = 10{,}000\, s_{\overline{10}|} + 500\left(\frac{\ddot{s}_{\overline{9}|} - 9}{.07}\right) = 138{,}164 + 27{,}260 = 165{,}424$$

The payment in the 10th year is equal to \$14,500 (the initial payment of \$10,000, plus 9 increases of \$500). This payment will increase by 3.5% for each of the final 10 years. Therefore, the total accumulation in the fund after 20 years is:

$$(165{,}424 \times 1.07^{10}) + (14{,}500 \times [(1.035)(1.07)^9 + (1.035)^2(1.07)^8 + \ldots + (1.035)^{10}])$$
$$= 325{,}414 + (14{,}500 \times 1.035^{10} \times [(1.07/1.035)^9 + (1.07/1.035)^8 + \ldots + 1])$$
$$= 325{,}414 + (20{,}454\, s_{\overline{10}|j}) \qquad \text{(where } j = 1.07/1.035 - 1 = .033816\text{)}$$
$$= 561{,}060$$

Answer is C.

Question 3

$\ddot{s}^{(m)}_{\overline{2n|}} = [(1 + i)^{2n} – 1]/d^{(m)} = [(1 + i)^{2n} – 1]/.08 = 180.24943 \Rightarrow (1 + i)^{2n} = 15.419954$

$\ddot{s}^{(m)}_{\overline{4n|}} = [(1 + i)^{4n} – 1]/d^{(m)} = [15.419954^2 – 1]/.08 = 2{,}959.69$

Answer is B.

Question 4

$d^{(4)} = .08$, and $d^{(4)}/4 = .02$

Recall that $i = \dfrac{d}{1-d}$.

$$i^{(4)}/4 = \frac{\frac{d^{(4)}}{4}}{1-\frac{d^{(4)}}{4}} = \frac{.02}{1-.02} = .020408$$

$i^{(12)}/12 = 1.020408^{1/3} - 1 = .006757$

Over the 20-year period, 240 monthly payments will be made. After the 43rd payment, there are 197 payments remaining. The present value immediately after the 43rd payment is:

$$a_{\overline{197|}.006757} = 108.72 = X$$

Let M equal the number of payments remaining at the point in time when the present value of the remaining payments is less than X/2 for the first time. Then,

$$a_{\overline{M|}.006757} = (1 – v^M)/.006757 = X/2 = 108.72/2 = 54.36$$

$\Rightarrow \quad v^M = .632689$

$\Rightarrow \quad M \times \ln(v) = \ln(.632689)$

$\Rightarrow \quad - 0.006734M = -0.457776$

$\Rightarrow \quad M = 67.98$

Therefore, there are 67 payments remaining when the present value is first less than X/2. That means that the number of payments already made is:

N = 240 – 67 = 173

Answer is E.

Question 5

First, let's consider Loan 1. The quarterly effective rate of interest is:

$i^{(4)}/4 = 1.08^{1/4} - 1 = .019427$

There are 20 quarterly payments over the 5 years. Each payment is:

$\$10{,}000 / a_{\overline{20|}.019427} = \$608.19$

A = total repayments = $608.19 × 20 = $12,163.80

Next, consider Loan 2. The monthly effective rate of interest is:

$i^{(12)}/12 = 1.08^{1/12} - 1 = .006434$

The monthly interest payment is:

$10,000 × .006434 = $64.34

The annual sinking fund payment is:

$\$10{,}000 / \ddot{s}_{\overline{4|}.09} = \$2{,}006.13$

B = total interest payments plus sinking fund payments
= ($64.34 × 48) + ($2,006.13 × 4) = $11,112.84

A – B = $12,163.80 - $11,112.84 = $1,050.96

Answer is D.

Question 6

Recall the following formula for a geometric series (which will be used below in the determination of the values of A and B):

$$1 + x + x^2 + \ldots + x^{n-1} = \frac{1 - x^n}{1 - x}$$

Annual payment = $\$10{,}000 / a_{\overline{40|}} = \$750.09$

The interest in a payment is equal to the outstanding balance immediately after the preceding payment, multiplied by the interest rate of 7%. After n payments, this is:

$\$750.09\, a_{\overline{40-n}|} \times .07 = \$750.09 \times [(1 - v^{40-n})/.07] \times .07 = \$750.09 \times (1 - v^{40-n})$

Therefore, we can conclude that the interest portion of a payment is equal to the payment multiplied by $(1 - v^{m})$, where m represents the number of remaining payments.

Similarly, since any payment consists of either interest or principal repayment, the principal portion of a payment is equal to the payment multiplied by v^{m}, where m represents the number of remaining payments.

Consider the even payments. After payment 1, there are 39 payments remaining. After payment 3, there are 37 payments remaining. And so on. In other words, including payment 2 there are 39 payments left, etc. The value of A is:

$$\begin{aligned} A &= \$750.09 \times [(1 - v^{39}) + (1 - v^{37}) + \ldots + (1 - v)] \\ &= \$750.09 \times [20 - (v^{39} + v^{37} + \ldots + v)] \\ &= \$750.09 \times [20 - v(1 + v^{2} + \ldots + v^{38})] \\ &= \$750.09 \times [20 - v(\frac{1 - v^{40}}{1 - v^{2}})] \\ &= \$9{,}833 \end{aligned}$$

Next, consider the odd payments. Including payment 1, there are 40 payments remaining. After payment 2, there are 38 payments remaining. And so on. The value of B is:

$$\begin{aligned} B &= \$750.09 \times [v^{40} + v^{38} + \ldots + v^{2})] \\ &= \$750.09 \times (\frac{v^{2} - v^{42}}{1 - v^{2}}) \\ &= \$4{,}831 \end{aligned}$$

$A + B = \$9{,}833 + \$4{,}831 = \$14{,}664$

Answer is D.

Question 7

Since Smith makes a 20% down payment, the amount of the mortgage is \$96,000 (\$120,000 × 80%).

$i^{(12)}/12 = .08/12 = .006667$

The initial monthly mortgage payment is:

$\$96{,}000/a_{\overline{360}|.006667} = \$704.41$

The interest in a payment is equal to the outstanding balance immediately after the preceding payment, multiplied by the interest rate of .6667%. After n payments, this is:

$\$704.41\, a_{\overline{360-n}|} \times .07 = \$704.41 \times [(1 - v^{360\text{-}n})/.07] \times .07 = \$704.41 \times (1 - v^{360\text{-}n})$

Therefore, we can conclude that the interest portion of a payment is equal to the payment multiplied by $(1 - v^m)$, where m represents the number of remaining payments. Note that there are 261 payments remaining immediately before the 100th payment is made.

The interest in the 100th payment is:

$A = \$704.41 \times (1 - v^{261}) = \$580.06$

The outstanding balance after 180 payments is:

$\$704.41 \times a_{\overline{180}|.006667} = \$73{,}710$

Using the new interest rate, $i^{(12)}/12 = .075/12 = .00625$

The revised monthly mortgage payment (with 120 payments remaining) is:

$\$73{,}710/a_{\overline{120}|.00625} = \$874.95$

Since any payment consists of both interest and principal repayment, the principal portion of a payment is equal to the payment multiplied by v^m, where m represents the number of remaining payments. Including the 100th payment of the refinanced mortgage, there are 21 remaining payments. The principal in the 100th refinanced payment is:

$B = \$874.95 \times v^{21} = \$767.64$

$A + B = \$580.06 + \$767.64 = \$1{,}347.70$

Answer is C.

Question 8

The price of the bond is equal to the present value of the future redemption payments plus the present value of the future coupon payments, all valued at a 7% yield rate.

The present value of the redemption payments is:

$$\begin{aligned}
PV_{redemption} &= 20{,}000v^{10} + 19{,}000v^{11} + \ldots + 1{,}000v^{29} \\
&= v^9 \times (20{,}000v + 19{,}000v^2 + \ldots + 1{,}000v^{20}) \\
&= v^9 \times (1{,}000 \times (Da)_{\overline{20|}}) \\
&= v^9 \times (1{,}000 \times \frac{20 - a_{\overline{20|}}}{.07}) \\
&= 73{,}089
\end{aligned}$$

The coupons are paid semiannually. Each coupon is 3% of the face amount. So, the semiannual coupon, payable for 10 years (20 payments), associated with the bond that will be redeemed after 10 years is \$600 (3% of \$20,000). The semiannual coupon, payable for 11 years (22 payments), associated with the bond that will be redeemed after 11 years is \$570 (3% of \$19,000). And so on through the semiannual coupon, payable for 29 years (58 payments), associated with the bond that will be redeemed after 29 years, which is \$30 (3% of \$1,000). The interest rate used must be a semi-annual rate:

$i^{(2)}/2 = 1.07^{1/2} - 1 = .034408$

The present value of the coupon payments, using an interest rate of 3.4408%, is:

$$\begin{aligned}
PV_{coupon} &= 600\,a_{\overline{20|}} + 570\,a_{\overline{22|}} + \ldots + 30\,a_{\overline{58|}} \\
&= 600(\frac{1-v^{20}}{i}) + 570(\frac{1-v^{22}}{i}) + \ldots + 30(\frac{1-v^{58}}{i}) \\
&= (30/.034408) \times [20(1-v^{20}) + 19(1-v^{22}) + \ldots + 1(1-v^{58})] \\
&= (30/.034408) \times [(20 + 19 + \ldots + 1) - (20v^{20} + 19v^{22} + \ldots + v^{58})] \\
&= (30/.034408) \times [\frac{20 \times 21}{2} - v^{18}(20v^2 + 19v^4 + \ldots + v^{40})]
\end{aligned}$$

Note that $v^2 = (1/1.034408)^2 = 1/1.07$. Therefore, the present value can be revised as follows, where the interest rate is 7%:

$$\begin{aligned}
&= (30/.034408) \times [\frac{20 \times 21}{2} - v^9(20v + 19v^2 + \ldots + v^{20})] \\
&= (30/.034408) \times [\frac{20 \times 21}{2} - v^9\,(Da)_{\overline{20|}}] \\
&= (30/.034408) \times [\frac{20 \times 21}{2} - v^9(\frac{20 - a_{\overline{20|}}}{.07})] \\
&= 119{,}371
\end{aligned}$$

The total price is the sum of the redemption value and the coupon value:

Price = 73,089 + 119,371 = 192,460

Answer is E.

Note in the equation above used to determine the present value of the coupons the use of the following summation formula:

$$1 + 2 + 3 + \ldots + n = \frac{n \times (n+1)}{2}$$

Question 9

The increase in the book value during the third year is equal to the difference between the present value of the future payments after the third year (after 6 coupon payments have been made) and the present value of the future payments after the second year (after 4 coupon payments have been made).

The coupons are made semi-annually, so each coupon is equal to \$45 (4.5% of the face amount of the bond).

The semi-annual yield rate is:

$$i^{(2)}/2 = 1.1025 - 1 = .05$$

The present value after 2 years (using the semi-annual yield rate of 5%) is:

$$PV_{2\text{ years}} = 1{,}050v^{16} + 45\, a_{\overline{16|}} = 968.72$$

The present value after 3 years (using the semi-annual yield rate of 5%) is:

$$PV_{3\text{ years}} = 1{,}050v^{14} + 45\, a_{\overline{14|}} = 975.76$$

The increase is: 975.76 – 968.72 = 7.04

Answer is B.

Question 10

The monthly interest rate is:

$i^{(12)}/12 = .07/12 = .005833$

The monthly payment on the loan is:

$\text{Payment} = \$100{,}000/\,a_{\overline{240}|.005833} = \$775.30$

The monthly yield rate for the investor is:

$i^{(12)}/12 = .08/12 = .006667$

The easiest way to solve this question is to calculate the present value of the future payments using the investor's yield rate for each of the given dates. Note that the amount to be repaid by the investor on each date is equal to the present value of the future loan payments valued using the loan interest rate.

If the loan is redeemed on 1/1/2010, there will have been 8 full years of loan payments from 1/1/2002 through 12/31/2009, for a total of 96 loan payments. The amount to be repaid on 1/1/2010 is equal to the present value of the remaining 10 years of loan payments (120 in all) using the loan interest rate.

$$PV_{1/1/2010\ \text{redemption}} = 775.30\,a_{\overline{96}|.006667} + 775.30\,a_{\overline{120}|.005833}\ v^{96}_{.006667} = 90{,}125$$

If the loan is redeemed on 1/1/2011, there will have been 9 full years of loan payments from 1/1/2002 through 12/31/2010, for a total of 108 loan payments. The amount to be repaid on 1/1/2011 is equal to the present value of the remaining 9 years of loan payments (108 in all) using the loan interest rate.

$$PV_{1/1/2011\ \text{redemption}} = 775.30\,a_{\overline{108}|.006667} + 775.30\,a_{\overline{108}|.005833}\ v^{108}_{.006667} = 89{,}798$$

Therefore, since the present value falls below the $90,000 price for the loan on 1/1/2011, the latest full repayment date for which the investor will yield at least 8% is 1/1/2010.

Answer is B.

Question 11

First, solve for S1. $i^{(4)}/4 = .08/4 = .02$ and $i^{(12)}/12 = 1.02^{1/3} - 1 = .006623$

$S1 = \$500\, s_{\overline{12}|.006623} = \$6{,}223$

Next, solve for A1. $d^{(2)}/2 = .06/2 = .03$

Recall that $i = \dfrac{d}{1-d}$.

$$i^{(2)}/2 = \frac{\frac{d^{(2)}}{2}}{1-\frac{d^{(2)}}{2}} = \frac{.03}{1-.03} = .030928$$

$A1 = \$6{,}223\, v^2_{.030928} = \$5{,}855$

Next, solve for S2. $d^{(12)}/12 = .06/12 = .005$

$$i^{(12)}/12 = \frac{\frac{d^{(12)}}{12}}{1-\frac{d^{(12)}}{12}} = \frac{.005}{1-.005} = .005025 \text{ and } i^{(4)}/4 = 1.005025^3 - 1 = .015151$$

$S2 = \$1{,}500\, s_{\overline{4}|.015151} = \$6{,}138$

Finally, solve for A2.

$A2 = A1 = \$5{,}855 = \$6{,}138v \Rightarrow \quad \$5{,}855 = \$6{,}138 \times 1/(1+i) \quad \Rightarrow \quad i = .048335$

Since P is convertible once every two years, $P = [(1.048335)^2 - 1]/2 = .049503$

P% = 4.9503%

Answer is E.

Question 12

The moment estimate of q_x is equal to the determination of q_x taking into account the actual exposure of the lives age x until death during the observation period (in this case between ages x and x+1). The fact that the underlying survival distribution is Balducci produces the following formula for ${}_tq_x$, where t < 1:

$${}_tq_x = \frac{t \cdot q_x}{1-(1-t)q_x} \qquad \text{(see page 66 of "Survival Models and Their Estimation")}$$

From the given information, there are 23 deaths between age x and x+1, and of the 100 lives in the observation group, 40 are exposed to death during only the first ¾ of the year. The remaining 60 are exposed to death for the entire year. The following equation can be written:

$$60q_x + 40\,{}_{\frac{3}{4}}q_x = 23$$

Applying the Balducci formula from above,

$$60q_x + 40\left(\frac{\frac{3}{4} \cdot q_x}{1-(1-\frac{3}{4})q_x}\right) = 23$$

$$60q_x + \frac{120 \cdot q_x}{4 - q_x} = 23$$

$$60q_x^2 - 383q_x + 92 = 0$$

Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, a = 60, b = -383, and c = 92.

Substituting,

$$q_x = \frac{-(-383) \pm \sqrt{(-383)^2 - (4)(60)(92)}}{(2)(60)} = .25$$

Answer is B.

Question 13

Recall the following approximations to the force of mortality (see formulas 1.19b and 1.20 in the "Life Contingencies" text):

$$\mu_x = (d_{x-1} + d_x)/2l_x = (l_{x-1} - l_{x+1})/2l_x \quad \text{and} \quad \mu_x = \tfrac{1}{2}(\log l_{x-1} - \log l_{x+1})$$

Therefore, in mortality table A,

$$\mu_{41} = (l_{40} - l_{42})/2l_{41} = (14{,}400 - 14{,}036)/(2 \times 14{,}219) = .0128$$

This is also the force of mortality at all ages in mortality table B. In mortality table B,

$$\mu_{44} = \tfrac{1}{2}(\log l_{43} - \log l_{45}) = \tfrac{1}{2}(\log l_{43} - \log(100{,}000)) = .0128 \quad \Rightarrow \quad l_{43} = 102{,}593$$

and $$\mu_{42} = \tfrac{1}{2}(\log l_{41} - \log l_{43}) = \tfrac{1}{2}(\log l_{41} - \log(102{,}593)) = .0128 \quad \Rightarrow \quad l_{41} = 105{,}253$$

Answer is C.

Question 14

Recall the following formulas under the assumption of uniform distributions of death (note that they can be found on page 66 of the text "Survival Models and Their Estimation"):

For $0 \le s \le 1$,

$${}_s p_x = 1 - s \times q_x$$
$$\mu_{x+s} = q_x/(1 - s \times q_x)$$

In addition, following the principles from page 59 of that text, for $0 \le r \le 1$ and $0 \le s \le 1$,

$${}_r q_{x+s} = 1 - {}_r p_{x+s} = 1 - \frac{l_{x+r+s}}{l_{x+s}} = 1 - \frac{l_x - (r+s) \times d_x}{l_x - s \times d_x} = \frac{l_x - s \times d_x}{l_x - s \times d_x} - \frac{l_x - (r+s) \times d_x}{l_x - s \times d_x}$$
$$= \frac{r \times d_x}{l_x - s \times d_x} = \frac{r \times q_x}{1 - s \times q_x}$$

Using the given data,

$${}_{0.5}q_{40.4} = \frac{.5 \times q_{40}}{1 - .4 \times q_{40}} = 0.025 \quad \Rightarrow \quad q_{40} = .049020$$
$${}_{0.9}p_{41} = 1 - .9 \times q_{41} = 0.955 \quad \Rightarrow \quad q_{41} = .05$$
$$\mu_{42.2} = q_{42}/(1 - .2 \times q_{42}) = 0.05 \quad \Rightarrow \quad q_{42} = .049505$$

Finally,

$_3p_{40} = l_{43}/l_{40} = p_{40} \times p_{41} \times p_{42} \Rightarrow \quad 100{,}000/l_{40} = (1 - q_{40}) \times (1 - q_{41}) \times (1 - q_{42})$

$\Rightarrow \quad 100{,}000/l_{40} = (1 - .049020) \times (1 - .05) \times (1 - .049505)$

$\Rightarrow \quad l_{40} = 116{,}454$

Answer is B.

Question 15

The present value of the life annuity normal form of payment is:

$$20{,}000\,\ddot{a}_{65} = 20{,}000 \times 10.3316 = 206{,}632$$

This is equal to the present value of the optional form of payment:

$$206{,}632 = X\ddot{a}_{65} + p_{65}q_{66}v^2\,(X + .95Xv + \ldots + X(.95v)^9)$$

Recall the geometric series $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{1-x^n}{1-x}$.

$$206{,}632 = 10.3316X + (.9887)(1 - .9873)v^2 \times [X(\frac{1-(.95v)^{10}}{1-.95v})]$$

$$X = 19{,}869$$

Answer is E.

Question 16

Recall from formula 3.49a on page 55 of the text "Survival Models and Their Estimation":

$_nm_x = {_nd_x}/{_nL_x}$

So,

$_2m_{35} = {_2d_{35}}/{_2L_{35}} = (d_{35} + d_{36})/(L_{35} + L_{36}) = (300 + d_{36})/(9{,}851 + 9{,}456) = .0404$

$\Rightarrow \quad d_{36} = 480 \quad \Rightarrow \quad l_{37} = 9{,}700 - 480 = 9{,}220$

From page 66 of the above text, for a constant force of mortality,

$l_{x+s} = l_x \times (p_x)^s$

Therefore,

$l_{36.5} = l_{36} \times (p_{36})^{.5} = 9{,}700 \times (9{,}220/9{,}700)^{.5} = 9{,}457$

Answer is D.

Question 17

From first principles,

$$
\begin{aligned}
e_{70:\overline{15}|} &= p_{70} + {}_2p_{70} + \ldots + {}_5p_{70} + {}_6p_{70} + \ldots + {}_{15}p_{70} \\
&= p_{70} + {}_2p_{70} + \ldots + {}_5p_{70} + {}_5p_{70}(p_{75} + \ldots + {}_{10}p_{75}) \\
&= e_{70:\overline{5}|} + {}_5p_{70}\, e_{75:\overline{10}|}
\end{aligned}
$$

$\Rightarrow \quad 11.45220 = 4.66234 + {}_5p_{70} \times 7.70883 \quad \Rightarrow \quad {}_5p_{70} = .88079$

Similarly,

$$
\begin{aligned}
e_{75:\overline{10}|} &= p_{75} + {}_2p_{75} + \ldots + {}_5p_{75} + {}_6p_{75} + \ldots + {}_{10}p_{75} \\
&= p_{75} + {}_2p_{75} + \ldots + {}_5p_{75} + {}_5p_{75}(p_{80} + \ldots + {}_5p_{80}) \\
&= e_{75:\overline{5}|} + {}_5p_{75}\, e_{80:\overline{5}|}
\end{aligned}
$$

$\Rightarrow \quad 7.70883 = 4.43230 + {}_5p_{75} \times 7.08531 \quad \Rightarrow \quad {}_5p_{75} = .802027$

Finally,

$$
\begin{aligned}
e_{70} &= p_{70} + {}_2p_{70} + \ldots + {}_5p_{70} + {}_6p_{70} + {}_7p_{70} + \ldots \\
&= p_{70} + {}_2p_{70} + \ldots + {}_5p_{70} + {}_5p_{70}(p_{75} + {}_2p_{75} + \ldots + {}_5p_{75} + {}_6p_{75} + \ldots) \\
&= e_{70:\overline{5}|} + {}_5p_{70}\,(e_{75:\overline{5}|} + {}_5p_{75}e_{80}) \\
&= 4.66234 + (.88079)[4.43230 + (.802027)(8.26871)] \\
&= 14.407428
\end{aligned}
$$

Answer is B.

Question 18

The number of people who die between ages 60 and 80 is $l_{60} - l_{80}$. The total lifetime after age 60 of those people who die between ages 60 and 80 is $T_{60} - T_{80} - 20l_{80}$. Therefore, the average number of years of future lifetime of those who die between age 60 and age 80 is:

$$N = \frac{T_{60} - T_{80} - 20l_{80}}{l_{60} - l_{80}}$$

Dividing numerator and denominator by l_{60},

$$N = \frac{(T_{60}/l_{60}) - (T_{60} \times {}_{20}p_{60}/l_{60}) - (20 l_{80}/l_{60})}{1 - l_{80}/l_{60}} = \frac{\mathring{e}_{60} - \mathring{e}_{60}\ {}_{20}p_{60} - 20\ {}_{20}p_{60}}{1 - {}_{20}p_{60}}$$

Since the force of mortality is a constant .1 at all ages, exactly 10% of the population is expected to die at each age. The complete life expectancy at any age is 10 years. Therefore, $\mathring{e}_{60} = 10$.

From formula 3.68 on page 62 of the text "Survival Models and Their Estimation" (in situations where there is a constant force of mortality),

$p_x = e^{-\mu} = e^{-.1} = .904837$

Therefore, ${}_{20}p_{60} = .904837^{20} = .135334$

Substituting,

$$N = \frac{10 - (10 \times .135334) - (20 \times .135334)}{1 - .135334} = 6.87$$

Answer is A.

Question 19

$i^{(12)}/12 = 1.07^{1/12} - 1 = .005654$
$d^{(12)}/12 = [(i^{(12)}/12)/(1 + i^{(12)}/12)] = .005622$
$d^{(12)} = .005622 \times 12 = .067464$

$\ddot{a}^{(12)}_{\overline{5}|} = (1 - v^5)/d^{(12)} = (1 - .712986)/.067464 = 4.2543$

Setting the present value of Option 1 equal to the present value of Option 2:

$X\ddot{a}^{(12)}_{65} = Y(\ddot{a}^{(12)}_{\overline{5}|} + {}_{5|}\ddot{a}^{(12)}_{65}) \Rightarrow 10.0833X = Y(4.2543 + 6.0553)$

$\Rightarrow X/Y = 1.0224$

Setting the present value of Option 1 equal to the present value of Option 3:

$X\ddot{a}_{65}^{(12)} = Y\ddot{a}_{65:65}^{(12)} + X(\ddot{a}_{65}^{(12)} - \ddot{a}_{65:65}^{(12)}) + (P/100)(Y)(\ddot{a}_{65}^{(12)} - \ddot{a}_{65:65}^{(12)})$
$10.0833X = 9.5833Y + X(10.0833 – 9.5833) + (P/100)(Y)(10.0833 – 9.5833)$
$10.0833(X/Y) = 9.5833 + (X/Y)(10.0833 – 9.5833) + (P/100)(10.0833 – 9.5833)$
$10.0833(1.0224) = 9.5833 + (1.0224)(10.0833 – 9.5833) + (P/100)(10.0833 – 9.5833)$
$P/100 = .4293$
$P = 42.93\%$

Answer is B.

Question 20

$p_{40}^{(T)} = p_{40}^{\prime(1)} \times p_{40}^{\prime(2)} = (1 -.05) \times (1 - .10) = .855$
$\ell_{41}^{(T)} = \ell_{40}^{(T)} \times p_{40}^{(T)} = 10{,}000 \times .855 = 8{,}550$
$p_{41}^{(T)} = \ell_{42}^{(T)} / \ell_{41}^{(T)} = 7{,}000/8{,}550 = .818713$ and $p_{41}^{(T)} = p_{41}^{\prime(1)} \times p_{41}^{\prime(2)} = (1 -.06) \times p_{41}^{\prime(2)}$
$\Rightarrow \quad (1 -.06) \times p_{41}^{\prime(2)} = .818713 \quad \Rightarrow \quad p_{41}^{\prime(2)} = .870971$
$\Rightarrow \quad q_{41}^{\prime(2)} = .129029$

Since $q_{41}^{\prime(1)}$ and $q_{41}^{\prime(2)}$ are both linear (uniformly distributed) over the interval [41,42], the following equality holds (this is the case from formula 10.6.3 in the “Actuarial Mathematics” text, where there are two decrements):

$q_{41}^{(1)} = q_{41}^{\prime(1)} \times (1 -$ 🗎 $q_{41}^{\prime(2)})$

So, $q_{41}^{(1)} = .06 \times (1 – (1/2)(.129029)) = .056129$

$d_{41}^{(1)} = \ell_{41}^{(T)} \times q_{41}^{(1)} = 8{,}550 \times .056129 = 480$

Answer is A.

Question 21

Recall the geometric series $1 + x + x^2 + \ldots = \dfrac{1}{1-x}$.

$$\ddot{a}_{72} = \ddot{a}_{75} = 1 + v p_{72} + v^2{}_2p_{72} + \ldots = 1 + .96v + (.96v)^2 + \ldots = \frac{1}{1-.96v} = 9.7273$$

$$_{10|}\ddot{a}_{72} = v^{10}{}_{10}p_{72} + v^{11}{}_{11}p_{72} + \ldots = v^{10}{}_{10}p_{72}\,(1 + v p_{82} + \ldots) = (.96v)^{10}\,(1 + .96v + \ldots)$$

$$= (.96v)^{10}\left(\frac{1}{1-.96v}\right) = 3.2875$$

$$\ddot{a}_{72:75} = 1 + v p_{72} p_{75} + v^2{}_2p_{72}\,{}_2p_{75} + \ldots = 1 + .96^2 v + (.96^2 v)^2 + \ldots = \frac{1}{1-.96^2 v} = 7.2102$$

$$\ddot{a}_{\overline{10|}} = (1 - v^{10})/.07 \times 1.07 = 7.5152$$

Form 1 Present Value $= 100\,\ddot{a}_{\overline{10|}} + 100\,{}_{10|}\ddot{a}_{72} = 100 \times (7.5152 + 3.2875) = 1{,}080.27$

Form 2 Present Value $= X\ddot{a}_{72:75} + 1.1X(\ddot{a}_{72} - \ddot{a}_{72:75}) + .5X(\ddot{a}_{75} - \ddot{a}_{72:75})$
$= 7.2102X + 1.1X(9.7273 - 7.2102) + .5X(9.7273 - 7.2102)$
$= 11.2376X$

Setting the actuarially equivalent present values equal to each other:

$11.2376X = 1{,}080.27 \qquad \Rightarrow \qquad X = 96.13$

Answer is C.

Question 22

Using the given values,

$_{30}p_{25} = p_{25} \times {}_{29}p_{26} = .99877 \times .90450 = .90339$

$A^{\ 1}_{25:\overline{30|}} = A_{25} - {}_{30}p_{25} v^{30} A_{55} = .0816496 - (.90339)(.17411)(.3051431) = .03365$

$A^{\ 1}_{25:\overline{40|}} = A_{25} - {}_{40}p_{25} v^{40} A_{65} = .0816496 - (.78766)(.09722)(.4397965) = .04800$

The present value of the benefits must be equal to the present value of the premiums.

$$200{,}000\, A^{\;1}_{25:\overline{30|}} + X\left(A^{\;1}_{25:\overline{40|}} - A^{\;1}_{25:\overline{30|}}\right) = 300 + 600 p_{25} v + 1{,}200\,{}_2p_{25} v^2 + 2{,}400\,{}_3p_{25} v^3 + 4{,}800\,{}_4p_{25} v^4 + 4{,}000\,{}_{30}p_{25} v^{30}$$

$$(200{,}000)(.03365) + X(.04800 - .03365) = 300 + (600)(.99877)(.94340) + (1{,}200)(.99877)(.99873)(.89000) + (2{,}400)(.99877)(.99873)(.99867)(.83962) + (4{,}800)(.99877)(.99873)(.99867)(.99861)(.79209) + (4{,}000)(.90339)(.17411)$$

$$6{,}730 + .01435X = 8349 \qquad \Rightarrow \qquad X = 112{,}822$$

Answer is B.

Question 23

Since the observation period is from time 3 to time r, the total period of time under observation is r-3. The average number of failures per year during that time period is equal to the ratio of the total period of time to the number of failures. Therefore, the average number of failures per year is $\frac{r-3}{3}$. Since failures are assumed to be uniform, it is expected that the failure of the remaining engine will occur at time r + $\frac{r-3}{3}$. There will no longer be any operational engines at that time. Therefore,

$$r + \frac{r-3}{3} = \omega = 13.67 \qquad \Rightarrow \qquad r = 11.0025$$

Answer is A.

Question 24

The equation of value for the life insurance policy is:

$$300 = 1{,}000\; {}_4p_{60} v^4 A_{64} \qquad \Rightarrow \qquad 300 = 1{,}000 \times (.998^4)(.762895)A_{64}$$
$$\Rightarrow \qquad A_{64} = .3964$$

Recall that $A_x = 1 - d\ddot{a}_x$. So,

$$A_{64} = 1 - d\ddot{a}_{64} \qquad \Rightarrow \qquad \ddot{a}_{64} = (1 - A_{64})/d = (1 - A_{64})/(i/(1+i))$$
$$= (1 - .3964)/(.07/1.07)$$
$$= 9.2265$$

Finally,

$$P = \ddot{a}_{\overline{3|}} + {}_3p_{60}v^3 + {}_4p_{60}v^4\,\ddot{a}_{64} = 2.8080 + (.998^3)(.816298) + (.998^4)(.762895)(9.2265)$$
$$= 10.60$$

Answer is C.

Question 25

Recall that s(x) = ${}_xp_0$ and s(x+1)/s(x) = ${}_{x+1}p_0/{}_xp_0 = p_x$.

Using first principles,

$$\ddot{a}_{80} = 1 + v\,p_{80} + v^2\,{}_2p_{80} + \ldots = 1 + \frac{s(81)}{s(80)}v + \frac{s(82)}{s(80)}v^2 + \ldots$$
$$= 1 + \frac{(20/101)}{(21/101)}v + \frac{(19/101)}{(21/101)}v^2 + \ldots + \frac{(1/101)}{(21/101)}v^{20}$$
$$= 1 + \frac{20}{21}v + \frac{19}{21}v^2 + \ldots + \frac{1}{21}v^{20}$$
$$= 1 + \frac{1}{21}(Da)_{\overline{20|}} = 1 + \frac{1}{21}\left(\frac{20 - a_{\overline{20|}}}{.05}\right) = 8.1788$$

Answer is B.

Question 26

$i^{(12)}/12 = 1.05^{1/12} - 1 = .004074 \qquad \Rightarrow \qquad i^{(12)} = .004074 \times 12 = .048888$

Since the load is 8% of the gross premium, the net premium is equal to 92% of the gross premium. Let G be the gross premium per dollar of benefit. The equation of value is:

$$.92G = a^{(12)}_{\overline{G|}} + {}_{G|}a^{(12)}_{65}$$

For G = 14,

$.92G = .92 \times 14 = 12.88 \qquad$ and $\qquad a^{(12)}_{\overline{14|}} + {}_{14|}a^{(12)}_{65} = 10.12 + 2.92 = 13.04$

The difference is 13.04 – 12.88 = .16

For G = 15,

$.92G = .92 \times 15 = 13.80$ and $a^{(12)}_{\overline{15|}} + {}_{15|}a^{(12)}_{65} = 10.62 + 2.56 = 13.18$

The difference is 13.18 – 13.80 = -.62

Therefore, the actual G must be between 14 and 15.

By interpolating,

$$G = 14 + \frac{.16}{.16 - (-.62)} = 14.2051$$

The annual payment is \$6,000. Therefore, the gross single premium is:

$6{,}000G = 6{,}000 \times 14.2051 = 85{,}231$

Answer is C.

Question 27

Since $\mu^{(1)}_x = \mu^{(2)}_x = (100 - x)^{-1}$, $0 < x < 1$, deaths must be uniformly distributed. In addition, there is one death per year for every 100 lives age 0.

Recall the following approximation that is true when deaths are distributed uniformly:

$$\begin{aligned}
{}_{10}q_0^{(1)} &= {}_{10}q_0'^{(1)}(1 - \text{[illegible]}\,{}_{10}q_0'^{(2)})/(1 - \text{[illegible]}\,{}_{10}q_0'^{(1)}\,{}_{10}q_0'^{(2)}) \\
&= \frac{10}{100}(1 - \text{[illegible]}(\frac{10}{100}))/(1 - \text{[illegible]}(\frac{10}{100})(\frac{10}{100})) \\
&= .0952
\end{aligned}$$

Answer is B.