Solutions to EA-1(A) Examination
Spring, 1997

Question 1

Step I: Calculate the time-weighted rate of return.

$$(1+\text{ time-weighted rate}) = \frac{\text{Balance on 7/1/97}}{\text{Balance on 1/1/97}} \times \frac{\text{Balance on date X}}{\text{Balance on 7/1/97*}}$$

$$\times \frac{\text{Balance on 12/31/97}}{\text{Balance on date X*}}$$

$$= \frac{1{,}030{,}000}{1{,}000{,}000} \times \frac{1{,}025{,}000}{1{,}030{,}000 - 50{,}000} \times \frac{1{,}150{,}000}{1{,}025{,}000 + 100{,}000}$$

$$= 1.1012$$

$$\Rightarrow \quad i = 10.12\%$$

* Balance after cash flow

Step II: Solve for the period of time interest is credited from date X in dollar weighted rate of return.

Ending Balance = (Beginning Balance)$(1 + i)$ + (Transaction)(1+fractional interest)

$1{,}150{,}000 = (1{,}000{,}000)(1 + i) - (50{,}000)(1 + ½i) + (100{,}000)(1 + ti)$,
where t represents the fraction of the year from the date of contribution to year-end

$1{,}150{,}000 = (1{,}000{,}000)(1.1012) - (50{,}000)(1 + ½(.1012)) + (100{,}000)(1 + t(.1012))$

Solving for t,

$t = .1314$

Note: The equation of value could have been written using compound interest. However, the result would not have been significantly different.

Step III: Determine date X.

The contribution was made with 13.14% of the year left. Since 13.14% of a year = 1.58 months, the contribution clearly must have been made during the month of November.

Answer is E.

Question 2

Key Concept: The total interest paid in the middle 12 payments is equal to the total payments less the principal repaid.

Total paid in the final 12 repayments = (100)(12 payments) = 1,200

Principal repaid in the final 12 repayments = 1,200 - 109.20 = 1,090.80

Therefore, the outstanding balance immediately prior to the last 12 payments is equal to 1,090.80.

So,

$100\, a_{\overline{12}|j} = 1{,}090.80$ where j = monthly effective interest rate

$a_{\overline{12}|j} = 10.9080$

Solving for j (by observation in the interest tables or using a calculator), $j = 1.5\%$.

Outstanding balance after 12 payments = $100\, a_{\overline{24}|.015} = 2{,}003.04$

Principal repaid in middle 12 installments = 2,003.04 - 1,090.80 = 912.24

Interest repaid in middle 12 installments = 1,200 - 912.24 = 287.76

Answer is B.

Question 3

Key Concept: The reduction in the total interest paid is equal to the difference between the payments that would have been made after 1/1/97 (had the loan not been refinanced) and the payments actually made. Note that in either case the principal repaid must be the same, so the difference must be interest.

Initial payment = $100{,}000 / a_{\overline{360}|.1/12} = 877.57$

Outstanding Balance on 1/1/97 = $877.57\, a_{\overline{300}|.1/12} = 96{,}574$

Revised payment (effective 1/1/97) = $96{,}574 / a_{\overline{180}|.09/12} = 979.52$

Total payments after 1/1/97 (original) = (877.57)(300 payments) = 263,271
Total payments after 1/1/97 (revised) = (979.52)(180 payments) = 176,314
Reduction in total interest = 263,271 - 176,314 = 86,957

Answer is C.

Question 4

Step I: Determine the amount of each repayment.

$$P = 200{,}000/\, a_{\overline{360}|.09/12} = 1{,}609.25$$

Step II: Determine the equivalent annual payment (including the additional repayment) if payments were made at the end of the year.

$$\text{Annual payment} = 1{,}609.25\, s_{\overline{12}|.09/12} + 1{,}609.25 = 21{,}737.08$$

Step III: Determine the year in which the loan will be paid off.

$$200{,}000 = 21{,}737.08\, a_{\overline{N}|i} \qquad \text{where } i = (1.0075)^{12} - 1 = .0938$$

N = 22.17 (by interpolating in the interest tables or using a calculator)

Note that $a_{\overline{22}|.0938} = 9.1779$, so that the present value of the first 22 years of payments is equal to $21{,}737.08 \times 9.1779 = 199{,}500.75$. Therefore, the final payment will be made in the 23rd year.

Step IV: Determine the exact number of payments in which the loan will be repaid.

The outstanding balance of the loan after 22 years is:

$$\text{Outstanding balance} = (200{,}000 - 199{,}500.75)(1.0938)^{22} = 3{,}588.83$$

The first two payments in the 23rd year will only total $1{,}609.25 \times 2 = 3{,}218.50$. Therefore, the loan will be repaid in the 3rd payment of the 23rd year for a total of $(22 \times 12) + 3$ payments, or 267 payments.

The reduction in the total number of payments is 360 - 267 = 93

Answer is D.

Question 5

Step I: Convert the discount rates to effective rates of interest.

Use the formula $i = d/(1 - d)$.

For the second 10 years in fund A, $i^{(4)}/4 = (d^{(4)}/4)/(1 - d^{(4)}/4) = (.09/4)/(1 - .09/4)$
$= .023018$

For the first 10 years in fund B, $i^{(12)}/12 = (d^{(12)}/12)/(1 - d^{(12)}/12) = (.09/12)/(1 - .09/12)$
$= .007557$

Step II: Solve for W and X.

Note that for the first 10 years in fund A, $i^{(4)}/4 = .06/4 = .015$
and for the second 10 years in fund B, $i^{(12)}/12 = .12/12 = .01$

$$W(1.015)^{40}(1.023018)^{40} = 4.5079W = Y$$

$$X(1.007557)^{120}(1.01)^{120} = 8.1455X = Z$$

It is given that W + X = 10,000. Multiplying both sides of the equation by 4.5079:

$$4.5079W + 4.5079X= 45,079$$

It is also given that Y + Z = 57,186. Substituting for Y and Z:

$$4.5079W+ 8.1455X = 57,186$$

Subtracting the first equation from the second equation:

$$(8.1455 - 4.5079)X= 57,186 - 45,079$$

$$X = 3,327$$

Substituting for X into either equation yields:

$$W = 6,673$$

Step III: Solve for Y.

Substituting into the first equation in step II:

$$Y = 4.5079W = (4.5079)(6,673) = 30,081$$

Answer is D.

Question 6

Key Concept: Examine the repayment flow to determine patterns that may be used to describe the payments.

The payment flow looks like:

100, 300, 500, 700, 900, 1100, 1300, 1500, 1700, 1900, 1600, 1300, 1000, 700, 400, 100.

The first and last payment are both \$100, so this can be viewed as three annuities:

1. A level annuity of \$100 paid at the end of each year for 16 years.
2. An arithmetic increasing annuity of \$200 paid at the end of each year for 9 years, beginning at the <u>end of the second year</u>. This can be thought of as an annuity immediate, discounted with interest for 1 year since the first payment is not due until the end of the second year.
3. An arithmetic decreasing annuity of \$300 paid at the end of each year for 5 years, beginning at the <u>end of the eleventh year</u>. Note that the payment in the final year has already been taken into account in the first annuity. This can be thought of as an annuity immediate, discounted with interest for 10 years since the first payment is not due until the end of the eleventh year.

The loan is equal to the present value of these payments.

$$
\begin{aligned}
\text{Loan} &= 100\, a_{\overline{16}|} + 200\,(Ia)_{\overline{9}|}\, v + 300\,(Da)_{\overline{5}|}\, v^{10} \\
&= 100\, a_{\overline{16}|} + 200[(\ddot{a}_{\overline{9}|} - 9v^{9})/i]v + 300[(5 - a_{\overline{5}|})/i]v^{10} \\
&= 945 + 5{,}543 + 1{,}961 \\
&= 8{,}449
\end{aligned}
$$

Answer is C.

Question 7

Step I: Determine the monthly payment.

Convert the annual effective rates of interest to monthly effective rates of interest.

For the first 15 years: $i_1^{(12)}/12 = (1.07)^{1/12} - 1 = .005654$

For the last 15 years: $i_2^{(12)}/12 = (1.11)^{1/12} - 1 = .008735$

An equation of value can be set up to solve for the monthly payment P:

$$1{,}000{,}000 = P\, a_{\overline{180}|.005654}\; a_{180\;\;.005654} + P\, a_{\overline{180}|.008735}\; v^{15}_{.07}$$

$\Rightarrow$ $P = 6{,}869$

Step II: Determine the interest in the 204th repayment.

The formula for determining interest in a particular repayment is:

Interest = $P(1 - v^{n})$ where P is the payment and n is the number of payments remaining including the current payment.

Since the total number of payments is 360, and the payment in question is the 204th,

$n = 360 - 203 = 157$

Note the use of 203 instead of 204 because the 204th payment must also be included in the count.

The monthly rate of interest to be used is .8735% since the period of time being discounted is during the last 15 years.

$$\text{Interest} = 6{,}869(1 - v_{.008735}^{157}) = 5{,}116$$

Answer is A.

Question 8

The formula for duration is:

$$\bar{d} = \frac{\sum_{t=1}^{n} tv^t R_t}{\sum_{t=1}^{n} v^t R_t}$$

(Where t = the time at which a payment is made and R_t = amount of payment at time t)

In this portfolio, $2,000 is payable on each 12/31 beginning on 12/31/2007 and ending on 12/31/2026, $10,000 is payable on 12/31/2001.

$$\bar{d} = \frac{[(5)(10{,}000v^5) + (11)(2{,}000v^{11}) + (12)(2{,}000v^{12}) + \ldots + (30)(2{,}000v^{30})]}{[10{,}000v^5 + 2{,}000v^{11} + 2{,}000v^{12} + \ldots + 2{,}000v^{30}]}$$

$$= \frac{[(5)(10{,}000v^5) + (10)(2{,}000a_{\overline{20|}}v^{10}) + (2{,}000v^{11}) + (2{,}000v^{12}) + \ldots + (2{,}000v^{30})]}{10{,}000v^5 + 2{,}000a_{\overline{20|}}v^{10}}$$

$$= \frac{50{,}000v^5 + 20{,}000a_{\overline{20|}}v^{10} + 2{,}000v^{10}(Ia)_{\overline{20|}}}{10{,}000v^5 + 2{,}000a_{\overline{20|}}v^{10}}$$

$$= 12.46$$

Note that this is the MacCaulay duration. The modified duration can be determined as follows:

$$\text{Modified duration} = \bar{d}/(1 + i)$$
$$= 12.46/1.08$$
$$= 11.54$$

Answer is C.

Question 9

Key Concept: The bond has been purchased at a discount since the amortized value is less than the face amount of the bond (and the maturity value of the bond). The difference between the amortized value as of consecutive coupon dates represents the write up of the bond. The write up can be thought of as principal repaid, and when the bond matures the amortized value will be exactly equal to $1,000.

Step I: Calculate the interest earned during the last half of 1997.

The write up in the value of the bond (principal repaid) during the last half of 1997 is equal to the difference between the amortized value (before coupon payment) at the beginning and end of the period.

Principal repaid = 943.78 - 939.33 = 4.45

The interest earned during the last half of 1997 is equal to the amount of the coupon plus the principal repaid. Note that the reason for adding the principal repaid is that purchasing the bond at a discount is really the same as a negative loan. Bonds purchased at a premium (the purchase price is greater than the face amount of the bond) act more like loans as we think of them, and, in the case of a bond purchased at a premium, the interest earned would equal the coupon less the principal repaid.

Since the coupon rate is 7% per year, payable semiannually, each coupon is equal to 3.5% of the $1,000 face amount, or $35.

Interest earned = 35.00 + 4.45 = 39.45

Step II: Calculate the yield rate.

The semiannual yield rate is:

$$i^{(2)}/2 = 39.45/(939.33 - 35.00) = .0436$$

Note that the $35 coupon is subtracted from the amortized value of the bond as of 6/30/97. This is necessary because the value of the bond is given *before* the coupon is paid. There is a cash flow of $35 on that date, reducing the value of the bond by the coupon paid.

The annual yield rate is:

$$i = (1 + i^{(2)}/2)^2 - 1 = 1.0436^2 - 1 = .0891\text{, or } 8.91\%.$$

Answer is E.

Question 10

The cash flow can be represented by payments of 1,3,6,10,15,21, etc. in each consecutive year. The cash flow can also be represented by the following chart.

1	**3**	**6**	**10**	**15**	**21**	**...**
1	2	3	4	5	6	...
	1	2	3	4	5	...
		1	2	3	4	...
			1	2	3	...
				1	2	...
					1	...

Note that the sum of the numbers under each column heading is equal to the column heading. Therefore, each row represents a payment pattern such that the sum of the payments in each row equals the total payment in the given perpetuity. Each row represents an increasing perpetuity due, with payments in each successive row beginning in each subsequent year.

The present value of this perpetuity can be written as:

$$\begin{aligned}
\text{Present value} &= (I\ddot{a})_{\overline{\infty}|} + v(I\ddot{a})_{\overline{\infty}|} + v^2(I\ddot{a})_{\overline{\infty}|} + \ldots \\
&= (I\ddot{a})_{\overline{\infty}|}\,(1 + v + v^2 + \ldots) \\
&= (I\ddot{a})_{\overline{\infty}|}\,\ddot{a}_{\overline{\infty}|} \\
&= [(1/i + 1/i^2)(1 + i)][(1/i)(1 + i)] \\
&= 125
\end{aligned}$$

Answer is D.

Question 11

Key Concept: The increases in each deposit occur annually, but the payments occur monthly. Determine an annual deposit equivalent to the monthly deposits so that the deposits and the increases occur over the same time period.

Step I: Determine the annual payment made on 1/1/81 that would be equivalent to the 1981 monthly deposits.

$$1/1/81 \text{ Payment} = 25\, a_{\overline{12}|.01} = 281.38$$

Note that the monthly effective rate of interest is 1% since the annual rate of interest compounded monthly is 12%.

Step II: Calculate the value of the savings account as of 1/1/99.

The annual effective rate of interest is:

$$i = 1.01^{12} - 1 = .126825$$

$$\begin{aligned}
\text{Accumulated value} &= (281.38)(1.126825)^{18} + [(281.38)(1.12)](1.126825)^{17} \\
&\quad + [(281.38)(1.12)^2](1.126825)^{16} + \ldots \\
&\quad + [(281.38)(1.12)^{17}](1.126825) \\
&= (281.38)(1.126825)^{18}[1 + (1.12/1.126825) + \ldots + (1.12/1.126825)^{17}] \\
&= (281.38)(1.126825)^{18}\,\ddot{a}_{\overline{18}|j} \\
&\qquad (\text{where } j = 1.126825/1.12 - 1 = .0060938) \\
&= 41{,}283
\end{aligned}$$

Answer is D.

Question 12

Key Concept: A pure endowment at age 65 is a payment at age 65 if the insured survives to that age.

The prospective formula for the terminal reserve is:

Reserve = Present value of future benefits less present value of future premiums.

In the case of this policy:

$$
\begin{aligned}
{}_{19}V &= 10{,}000\, A_{64:\overline{1}|} - 350\, \ddot{a}_{\overline{1}|} \\
&= 10{,}000[(C_{64} + D_{65})/D_{64}] - 350 \\
&= 10{,}000[((vD_{64} - D_{65}) + D_{65})/D_{64}] - 350 \\
&= 10{,}000v - 350
\end{aligned}
$$

Since the final premium is not paid, the present value of the future benefits must be equal to the reserve after 19 years. The future benefits consist of a death benefit payable at the end of the year if the insured dies between age 64 and 65, and an endowment paid if the insured survives to age 65.

$$
\begin{aligned}
&{}_{19}V = 10{,}000(C_{64}/D_{64}) + E(D_{65}/D_{64}) \qquad \text{(where E = the pure endowment)} \\
&10{,}000v - 350 = 10{,}000[(vD_{64} - D_{65})/D_{64}] + E(D_{65}/D_{64}) \\
&10{,}000v - 350 = 10{,}000v - 10{,}000(D_{65}/D_{64}) + E(D_{65}/D_{64})
\end{aligned}
$$

$\Rightarrow \quad E = 9{,}620$

Answer is B.

Question 13

Key Concept: The annuity is payable as long as either Smith is alive and eight years has not passed or Brown is alive and four years has not passed. This is basically a joint and 100% survivor annuity with a maximum number of payments. The annuity can be thought of as Smith being the primary annuitant and Brown being the beneficiary should Smith die. The payments to beneficiary Brown would only be made if four years have not passed.

The present value (net single premium) of the annuity can be written as:

$$PV = 1{,}000\, a_{10:\overline{8}|} + 1{,}000(a_{14:\overline{4}|} - a_{10:14:\overline{4}|})$$

Note that the subtraction in the beneficiary portion of the annuity is limited to a maximum of four years (not eight) since the annuity payable to Brown would only be paid during the first four years.

Evaluating the present value:

$$PV = 1{,}000(5.78 + 3.22 - 3.12) = 5{,}880$$

Answer is D.

Question 14

Key Concept: The probability of withdrawal before age 42 is equal to the probability of withdrawal at age 40 plus the probability of withdrawal at age 41.

Step I: Calculate the probability of withdrawal at age 40.

The total that leave the population while age 40 is:

$$d_{40}^{(T)} = l_{40}^{(T)} - l_{41}^{(T)} = 100{,}000 - 93{,}674 = 6{,}326$$

The total that leave due to withdrawal is equal to the total that leave less the number who die.

$$d_{40}^{(w)} = 6{,}326 - d_{40}^{(d)} = 6{,}326 – 213 = 6{,}113$$

The probability of withdrawal at age 40 is:

$$q_{40}^{(w)} = d_{40}^{(w)} / l_{40}^{(T)} = 6{,}113/100{,}000 = .061130$$

Step II: Calculate the probability of withdrawal between age 41 and 42.

The individual must first survive to age 41. The probability is:

$$p_{40}^{(T)} = l_{41}^{(T)} / l_{40}^{(T)} = 93{,}674/100{,}000 = .936740$$

Next calculate the probability that an individual currently age 41 withdraws before age 42. This probability is $q_{41}^{(w)}$. Since the data given at age 41 is for the *rate* of death instead of the *probability* of death, it will be necessary to convert the rates into probabilities.

$$p_{41}^{(T)} = l_{42}^{(T)} / l_{41}^{(T)} = 87{,}867/93{,}674 = .938008$$

and $$p_{41}^{(T)} = p_{41}^{'(d)} \times p_{41}^{'(w)} = (1 - q_{41}^{'(d)}) \times p_{41}^{'(w)} = .9976\, p_{41}^{'(w)}$$

$\Rightarrow$ $$.9976\; p_{41}^{'(w)} = .938008$$

$\Rightarrow$ $$p_{41}^{'(w)} = .940265$$

$\Rightarrow$ $$q_{41}^{'(w)} = 1 - p_{41}^{'(w)} = .059735$$

Recall the formula for converting from rates to probabilities:

$$q_x^{(w)} = [\, q_x^{'(w)}\ (1 - ½\, q_x^{'(d)})]/(1 - ¼\, q_x^{'(w)}\ q_x^{'(d)})$$

Note that the denominator is so close to one that it is insignificant for purposes of this problem. So, the formula reduces to:

$$q_x^{(w)} = q_x^{'(w)}\ (1 - ½\, q_x^{'(d)})$$

In the case of this problem:

$$q_{41}^{(w)} = q_{41}^{'(w)} \ (1 - ½ \, q_{41}^{'(d)}) = (.059735)[1 - ½(.0024)] = .059663$$

Step III: Calculate the total probability of withdrawal before age 42.

The total probability is:

$$_{2}q_{40}^{(w)} = q_{40}^{(w)} + p_{40}^{(T)} \ q_{41}^{(w)} = .061130 + (.936740)(.059663) = .117019$$

Answer is C.

Question 15

The total population is equal to the number of students in the population who entered at age 18 plus the number who entered at age 19. The commutation function T_x represents the number of people living at age *x* and older. Therefore, T_{18} - T_{22} represents the number of people between age 18 and 22, and T_{19} - T_{22} represents the number of people between age 19 and 22.

The number of students between age 18 and 22 who entered at age 18 is equal to:

$(4,000/l_{18})(T_{18} - T_{22}) = (4,000/100,000)(266,668 - 0) = 10,667$

Note that the number of people between ages 18 and 22 (T_{18} - T_{22}) must be multiplied by the ratio of the number of entrants at age 18 to l_{18}. This is necessary because the commutation functions represent relative values for a general population, and we must determine the ratio of the actual population, in this case the students, to the relative population.

The number of students between age 18 and 22 who entered at age 18 is equal to:

$(X/l_{19})(T_{19} - T_{22}) = (X/93,750)(168,751 - 0) = 1.8X$

Since the total population of 15,000 students is equal to the number of students in the population who entered at age 18 plus the number who entered at age 19:

15,000 = 10,667 + 1.8X

⇒ X = 2,407

Answer is B.

Question 16

If Smith is alive then there are two possibilities.

(1) Both Smith and Brown are alive.
(2) Only Smith is alive.

Since \$100 is the present value if Smith is alive and \$20 is the present value in case (2), then the present value in case (1) must be \$80.

If only Smith or only Brown is alive then there are two possibilities.

(1) Only Smith is alive.
(2) Only Brown is alive.

Since \$50 is the present value if only Smith or only Brown is alive and \$20 is the present value in case (1), then the present value in case (2) must be \$30.

If Brown is alive then there are two possibilities.

(1) Both Smith and Brown are alive.
(2) Only Brown is alive.

The value in case (1) is \$80 and the value in case (2) is \$30. Therefore the total value is:

$$X = 80 + 30 = 110$$

Answer is D.

Question 17

The present value of this annuity is:

$$PV = (50)(12\ddot{a}_{65}^{(12)}) + 10{,}000\,A^{1}_{65:\overline{10}|} = (600)(N_{65}^{(12)}/D_{65}) + (10{,}000)(M_{65} - M_{75})/D_{65}$$

Calculate the values of $N_{65}^{(12)}$, M_{65} and M_{75}.

$$N_{65}^{(12)} = N_{65} - (11/24)D_{65} = 8{,}872 - (11/24)(965) = 8{,}430$$

$$M_{65} = vN_{65} - N_{66} = vN_{65} - (N_{65} - D_{65}) = (1/1.07)(8{,}872) - (8{,}872 - 965) = 385$$

$$M_{75} = vN_{75} - N_{76} = vN_{75} - (N_{75} - D_{75}) = (1/1.07)(2{,}379) - (2{,}379 - 346) = 190$$

Substituting into the present value equation,

$$\begin{aligned} PV &= (600)(N_{65}^{(12)}/D_{65}) + (10{,}000)(M_{65} - M_{75})/D_{65} \\ &= (600)(8{,}430/965) + (10{,}000)(385 - 190)/965 \\ &= 7{,}262 \end{aligned}$$

Answer is B.

Question 18

Using the given formula $l_x = 1{,}000 - 10x$,

$$q_{[68]+2}/q_{68+2} = q_{[68]+2}/[(l_{70} - l_{71})/l_{70}] = q_{[68]+2}/[(300 - 290)/300] = .90$$

$\Rightarrow$ $q_{[68]+2} = .030000$

$$p_{[68]+2} = l_{[68]+3}/l_{[68]+2} = l_{68+3}/l_{[68]+2} = 290/l_{[68]+2}$$

and $p_{[68]+2} = 1 - q_{[68]+2}$

$\Rightarrow$ $1 - q_{[68]+2} = 290/l_{[68]+2}$

$\Rightarrow$ $1 - .03 = 290/l_{[68]+2}$

$\Rightarrow$ $l_{[68]+2} = 299$

Answer is D.

Question 19

Note that since q_{76} is equal to 1.00, the mortality table ends after age 76.

For ages less than or equal to age 35, $q_x = .01/1.01$ $\Rightarrow$ $p_x = 1 - q_x = 1 - .01/1.01 = 1/1.01$

For ages greater than age 35, $q_x = .02/1.02$ $\Rightarrow$ $p_x = 1 - q_x = 1 - .02/1.02 = 1/1.02$

So,

$$\begin{aligned} e_0 &= p_0 + {}_2p_0 + {}_3p_0 + \ldots + {}_{76}p_0 \\ &= p_0 + p_0p_1 + p_0p_1p_2 + \ldots + (p_0p_1\ldots p_{75}) \\ &= (1/1.01) + (1/1.01)^2 + \ldots + (1/1.01)^{36} + (1/1.01)^{36}(1/1.02) + \ldots + (1/1.01)^{36}(1/1.02)^{40} \\ &= a_{\overline{36}|.01} + (1/1.01)^{36}[(1/1.02) + (1/1.02)^2 + \ldots + (1/1.02)^{40}] \\ &= a_{\overline{36}|.01} + (1/1.01)^{36}\, a_{\overline{40}|.02} \\ &= 30.1075 + (.6989)(27.3555) \\ &= 49.2263 \end{aligned}$$

Answer is B.

Question 20

$1000\ {}_5V_{40} = 1000A_{45} - P\ddot{a}_{45}$ (where P = Net annual premium)

$\Rightarrow$ $43.46 = 1000(1 - d\ddot{a}_{45}) - P\ddot{a}_{45}$

$\Rightarrow$ $43.46 = 1000 - (1000d + P)\ddot{a}_{45}$

$\Rightarrow$ $43.46 = 1000 - (1000d + P)(N_{45}/D_{45})$

$\Rightarrow$ $(1000d + P) = 74.80$

$$\begin{aligned}
1000\ {}_6V_{40} &= 1000A_{46} - P\ddot{a}_{46} \\
&= 1000(1 - d\ddot{a}_{46}) - P\ddot{a}_{46} \\
&= 1000 - (1000d + P)\ddot{a}_{46} \\
&= 1000 - (1000d + P)(N_{46}/D_{46}) \\
&= 1000 - (74.80)((N_{45} - D_{45})/D_{46}) \\
&= 53.30
\end{aligned}$$

Answer is E.

Question 21

Step I: Determine the effective monthly rate of interest.

$D_{41}/D_{40} = vp_{40} = v(1 - q_{40}) = .997875v$

$\Rightarrow$ $607/651 = .997875v$

$\Rightarrow$ $v = .934397$

$\Rightarrow$ $i = .070209$

$i^{(12)}/12 = 1.070209^{1/12} - 1 = .005671$

Step II: Determine an expression for the present value of annuity A.

$PV_A = P\,a_{\overline{\infty}|.005671} = (P)(1/.005671) = 176.34P$

Step III: Determine the present value of annuity B.

$PV_B = 12{,}000\,a_{40}^{(12)} = 12{,}000[(N_{41} + (11/24)D_{40})/D_{40}] = 153{,}869$

Step IV: Solve for \$P.

The two annuities are actuarially equivalent, so the present values can be set equal to each other.

$176.34P = 153{,}869$

$\Rightarrow$ $P = 872.57$

Answer is B.

Question 22

Step I: Determine annual annuity values.

$\ddot{a}_x = \ddot{a}_x^{(12)} + 11/24 = 7.6022 + .4583 = 8.0605$

$\ddot{a}_{x+1} = \ddot{a}_{x+1}^{(12)} + 11/24 = 7.3683 + .4583 = 7.8266$

$\ddot{a}_{x+2} = \ddot{a}_{x+2}^{(12)} + 11/24 = 7.1321 + .4583 = 7.5904$

Step II: Determine probabilities of survival at ages x and x+1.

Recall the formula for successive annuities due:

$\ddot{a}_x = 1 + vp_x\, \ddot{a}_{x+1}$

Substituting into this formula:

$8.0605 = 1 + (1/1.07)(7.8266)p_x$

$\Rightarrow \quad p_x = .9653$

Similarly:

$\ddot{a}_{x+1} = 1 + vp_{x+1}\, \ddot{a}_{x+2}$

$\Rightarrow \quad 7.8266 = 1 + (1/1.07)(7.5904)p_{x+1}$

$\Rightarrow \quad p_{x+1} = .9623$

Step III: Determine ${}_{1|}q_x$.

${}_{1|}q_x = (p_x)(q_{x+1}) = (p_x)(1 - p_{x+1}) = (.9653)(1 - .9623) = .0364$

Answer is B.

Question 23

Key Concept: The present value of the net premiums is equal to the present value of the benefits.

Let P represent the net annual premium.

$P\,\ddot{a}_{40:\overline{25|}} = (12)(1{,}000)({}_{25|}\ddot{a}_{40}^{(12)})$

$P[(N_{40} - N_{65})/D_{40}] = (12{,}000)(N_{65}^{(12)}/D_{40})$

Note that $N_x = N_x^{(12)} + (11/24)D_x$. So,

$$\begin{aligned} N_{40} - N_{65} &= [\, N_{40}^{(12)} + (11/24)D_{40}] - [\, N_{65}^{(12)} + (11/24)D_{65}] \\ &= N_{40}^{(12)} - N_{65}^{(12)} + (11/24)(D_{40} - D_{65}) \\ &= 49{,}531 \end{aligned}$$

Substituting into the original equation:

$$(49{,}531/D_{40})P = (12{,}000)(\, N_{65}^{(12)}/D_{40})$$

$$11.1206P = 11{,}639$$
$$P = 1{,}046.62$$

Answer is B.

Question 24

Recall the identity: $e_x = p_x(1 + e_{x+1})$.

So,

$$e_{63} = p_{63}(1 + e_{64}) \quad \Rightarrow \quad 9.5 = p_{63}(1 + 9.0) \quad \Rightarrow \quad p_{63} = .95$$

$$e_{64} = p_{64}(1 + e_{65}) \quad \Rightarrow \quad 9.0 = p_{64}(1 + 8.5) \quad \Rightarrow \quad p_{64} = .947368$$

And,

$${}_2q_{63} = 1 - {}_2p_{63} = 1 - (.95)(.947368) = .1$$

Answer is B.

Question 25

Recall the formula for the present value of a whole life insurance:

$$\begin{aligned} A_x &= vq_x + v^2{}_{1|}q_x + v^3{}_{2|}q_x + v^4{}_{3|}q_x + v^5{}_{4|}q_x + \ldots \\ &= vq_x + v^2{}_{1|}q_x + v^3{}_{2|}q_x + v^3{}_3p_x(vq_{x+3} + v^2{}_{1|}q_{x+3} + \ldots) \\ &= vq_x + v^2{}_{1|}q_x + v^3{}_{2|}q_x + v^3{}_3p_xA_{x+3} \end{aligned}$$

The net single premium (or present value) of the whole life insurance at age 50 is:

$$\begin{aligned} 5{,}000 = 10{,}000A_{50} &= 10{,}000(vq_{50} + v^2{}_{1|}q_{50} + v^3{}_{2|}q_{50} + v^3{}_3p_{50}A_{53}) \\ &= 10{,}000(v[l_{50} - l_{51}]/l_{50} + v^2[l_{51} - l_{52}]/l_{50} + v^3[l_{52} - l_{53}]/l_{50} + v^3(l_{53}/l_{50})A_{53}) \\ &= 10{,}000(.046296 + .042867 + .039692 + .674757A_{53}) \end{aligned}$$

$$\Rightarrow \quad A_{53} = .550042$$

The net single premium at age 53 is:

$$10{,}000A_{53} = (10{,}000)(.550042) = 5{,}500$$

Answer is C.

Solutions to EA-1(A) Examination
Spring, 1998

Question 1

The annual rate of return is equal to i, where i is the effective annual rate of interest such that the value of the fund is equal to the accumulated value of the contributions:

$$14{,}000 = (35{,}000)(1 + i)^8 - (70{,}000)(1 + i)^4$$
$$\Rightarrow \quad 0 = (35{,}000)(1 + i)^8 - (70{,}000)(1 + i)^4 - 14{,}000$$

This can be written as:

$$0 = 35{,}000x^2 - 70{,}000x - 14{,}000 \quad (\text{where } x = (1 + i)^4)$$
$$= 5x^2 - 10x - 2$$

Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, a = 5, b = -10, and c = -2.

Substituting into the quadratic formula:

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - (4)(5)(-2)}}{(2)(5)} = 2.183216$$

Since $x = 2.183216$ and $x = (1 + i)^4$,

$$(1 + i)^4 = 2.183216$$
$$\Rightarrow \quad 1 + i = 1.2156$$
$$\Rightarrow \quad i = 21.56\%$$

Answer is C.

Test taking note: The correct range can be determined by using the interest rate at the endpoint of each range to determine which range must contain the correct effective rate of interest.

Using $i = 14\%$, $(35{,}000)(1.14)^8 - (70{,}000)(1.14)^4 = -18{,}387$
Using $i = 19\%$, $(35{,}000)(1.19)^8 - (70{,}000)(1.19)^4 = 375$
Using $i = 24\%$, $(35{,}000)(1.24)^8 - (70{,}000)(1.24)^4 = 30{,}138$

Since 14,000 is between 375 and 30,138, the effective rate of interest must be between 19% and 24%.

Question 2

The gain as a percentage rate of return for 1997 is equal to 1% (7% - 6%).

Writing the equation representing the dollar-weighted rate of return (reflecting only the gain):

$$\begin{aligned} \text{Gain} &= (60{,}000)(.01) + (30{,}000)(.01)(1/6) - (1000)(.01)(1 + 11/12 + \ldots + 1/12) \\ &= 600 + 50 - (10)(1/12)(12 + 11 + \ldots + 1) \\ &= 650 - (10)(1/12)[(12)(13)/2] \\ &= 585 \end{aligned}$$

Answer is A.

Question 3

The equation of value is:

$$\begin{aligned} 1{,}000{,}000 &= 5{,}500\, \ddot{s}_{\overline{15|}}\, (1.09)^{16} + X\, \ddot{s}_{\overline{16|}} \\ &= (5{,}500)(32.003399)(3.970306) + 35.973705X \end{aligned}$$

$$X = 8{,}371$$

Answer is C.

Question 4

Step I: Determine the effective monthly interest rate.

$$i^{(12)}/12 = 1.07^{1/12} - 1 = .005654$$

Step II: Determine the present value of the payments.

$$\begin{aligned} \text{Present value} &= 500\, \ddot{a}_{\overline{60|}}/1.07 + 650\, \ddot{a}_{\overline{60|}}/1.07^{6} + 10{,}000/1.07^{11} \\ &= (500)(51.048882)/1.07 + (650)(51.048882)/1.500730 + 10{,}000/2.104852 \\ &= 50{,}716 \end{aligned}$$

Answer is B.

Question 5

Step I: Determine the outstanding balance of the loan on 1/1/2007.

The outstanding balance is equal to the difference between the accumulated loan and the accumulated payments.

$$\begin{aligned} \text{Accumulated loan}_{1/1/2007} &= 10{,}000 \times 1.07^{9} \\ &= 10{,}000 \times 1.838459 \\ &= 18{,}385 \end{aligned}$$

$$\text{Accumulated payments}_{1/1/2007} = (1{,}000\, s_{\overline{5|}} \times 1.07^4) + (3{,}000 \times 1.07) + 1{,}000$$
$$= (1{,}000)(5.750739)(1.310796) + (3{,}000)(1.07) + 1{,}000$$
$$= 11{,}748$$

$$\text{Outstanding balance}_{1/1/2007} = \text{Accumulated loan}_{1/1/2007} + \text{Accumulated payments}_{1/1/2007}$$
$$= 18{,}385 - 11{,}748$$
$$= 6{,}637$$

Step II: Determine the interest paid in the year 2007.

The interest in the year 2007 repayment is equal to the outstanding balance at the beginning of the year times the annual interest rate of 7%.

$$\text{Interest}_{2007} = 6{,}637 \times .07 = 465$$

Answer is C.

Question 6

Key concept: The purchase price of a bond is equal to the present value of the redemption amount plus the present value of the future coupons. In the case of this serial bond, it is redeemed in level increments each year, and the coupons would decrease each year as each increment of the bond is redeemed. Note that a coupon is paid for an increment of the bond in the year in which the bond is redeemed.

Redemption amount per increment = 10,000 × 5% = 500

Coupon per increment = 10,000 × 5% × 7% = 35

$$\text{Price} = 500\, a_{\overline{20|}} + 35\,(Da)_{\overline{20|}}$$
$$= 500\, a_{\overline{20|}} + 35[(20 - a_{\overline{20|}})/i]$$
$$= (500)(7.469444) + (35)(104.421303)$$
$$= 7{,}389$$

Answer is A.

Question 7

$$\ddot{a}_{\overline{n+2|}} = a_{\overline{n+1|}} + 1 = 14.030 \quad \Rightarrow \quad a_{\overline{n+1|}} = 13.030$$

$$\ddot{s}_{\overline{n|}} = s_{\overline{n+1|}} - 1 = 52.344 \Rightarrow \quad s_{\overline{n+1|}} = 53.344$$

$$a_{\overline{n+1|}} = v^{n+1}\, s_{\overline{n+1|}} \quad \Rightarrow \quad v^{n+1} = a_{\overline{n+1|}} / s_{\overline{n+1|}} = 13.030/53.344 = .244264$$

$$(1 + i)^{n+1} = 1/v^{n+1} = 1/.244264 = 4.093931$$

$s_{\overline{n+1|}} = [(1 + i)^{n+1} - 1]/i \quad = (4.093931 - 1)/i$

$\Rightarrow \quad 53.344 = (4.093937 - 1)/i \quad \Rightarrow \quad i = .058$

Answer is E.

Question 8

Key concept: The principal repaid in a given payment is equal to the amount of the payment times v^n, where n is equal to the number of remaining payments (including the current payment). The principal repaid in a series of payments is equal to the sum of the principal repaid in the individual payments. The interest repaid is equal to the total payment less the principal repaid.

Let L = Initial loan amount

Note that the monthly effective rate of interest is $i^{(12)}/12 = .18/12 = .015$

$$\text{Payment} = L/a_{\overline{36|}.015} = L/27.660684 = .036152L$$

$$\begin{aligned}\text{Principal repaid} &= (.036152L)(v^{36} + v^{35} + ... + v^{13}) \\ &= (.036152L)(a_{\overline{36|}.015} - a_{\overline{12|}.015}) \\ &= (.036152L)(27.660684 - 10.907505) \\ &= .605661L\end{aligned}$$

$$\text{Interest repaid} = (24 \text{ payments})(.036152L) - .605661L = .261987L$$

$$\text{Ratio of interest to principal} = .261987L/.605661L = .432564$$

Answer is C.

Alternative Solution:

The principal repaid is equal to the difference between the original loan amount and the outstanding balance after the 12/31/97 payment.

Let P = Payment

$$\text{Original Loan} = P \times a_{\overline{36|}.015}$$

$$\text{Outstanding Balance}_{12/31/97} = P \times a_{\overline{12|}.015}$$

$$\begin{aligned}\text{Principal Repaid} &= P(a_{\overline{36|}.015} - a_{\overline{12|}.015}) \\ &= P(27.660684 - 10.907505) \\ &= 16.753179P\end{aligned}$$

$$\text{Interest Repaid} = 24P - 16.753179P = 7.246821P$$

$$\text{Ratio of interest to principal} = 7.246821P/16.753179P = .432564$$

Question 9

Step I: Determine the outstanding balance immediately after the seventh payment.

Outstanding balance = $10{,}000\, a_{\overline{23}|}$ = 112,722

Step II: Determine the combined loan payment.

Combined payment = $(112{,}722 + 50{,}000)/\, a_{\overline{14}|}$ = 18,606

Step III: Determine the interest portion of the second combined installment.

The interest portion of a payment is equal to the payment times ($1 - v^n$), where n is equal to the number of remaining payment including the payment in question. There are 13 remaining payments, including the 2nd combined payment.

Interest in 2nd payment = $(18{,}606)(1 - v^{13})$ = 10,885

Alternatively, the interest portion of the second payment is equal to the outstanding balance at the beginning of the second year times the interest rate.

Outstanding balance$_{\text{2nd yr}}$ = $18{,}606\, a_{\overline{13}|}$ = 155,502

Interest in 2nd payment = (155,502)(.07) = 10,885

Answer is C.

Question 10

Step I: Determine the present value of the scholarship.

The semiannual effective rate of interest must be calculated since the payments are made semiannually.

$i^{(2)}/2 = 1.08^{1/2} - 1 = .039230$

Since the increases in the tuition occur annually, calculate the present value as of 9/1 of one year's tuition.

PV = 10,000 × (1 + 1/1.039230) = 19,623

Note that since the annual tuition is $20,000, each semiannual payment is $10,000. It has now been determined that a payment of $19,623 payable on 9/1 each year is equivalent to two payments of $10,000 on 9/1 and the following 3/1.

The present value of the tuition can now be calculated. Note that the first tuition payment is not made for three years, but the tuition amount increases at the rate of 2.5% each year nevertheless.

$$\begin{aligned}
\text{PV of tuition} &= 19{,}623 \times [1.025^3/1.08^3 + 1.025^4/1.08^4 + ... + 1.025^6/1.08^6] \\
&= 19{,}623 \times 1.025^2/1.08^2 \times [(1.025/1.08) + (1.025/1.08)^2 + \\
&\qquad ... + (1.025/1.08)^4] \\
&= 19{,}623 \times \ddot{a}_{\overline{4}|j} \qquad \text{(where } j = 1.08/1.025 - 1 = .053659) \\
&= 62{,}147
\end{aligned}$$

Step II: Determine the present value of the perpetuity.

The 2.5% increases on the perpetuity payments do not begin until the perpetuity begins.

$$\begin{aligned}
\text{PV of perpetuity} &= 100{,}000 \times [1/1.08^7 + 1.025/1.08^8 + 1.025^2/1.08^9 + ...] \\
&= 100{,}000/1.08^7 \times [1 + 1.025/1.08 + (1.025/1.08^2) + ...] \\
&= 100{,}000/1.08^7 \times \ddot{a}_{\overline{\infty}|j} \qquad \text{(where } j = 1.08/1.025 - 1 = .053659) \\
&= 100{,}000/1.08^7 \times a_{\overline{\infty}|j} \times (1 + j) \\
&= 100{,}000/1.08^7 \times 1/j \times (1 + j) \\
&= 1{,}145{,}763
\end{aligned}$$

Step III: Determine the total present value.

Total present value = 62,147 + 1,145,763 = 1,207,910

Answer is D.

Question 11

The formula for duration is:

$$\bar{d} = \frac{\sum_{t=1}^{n} tv^t R_t}{\sum_{t=1}^{n} v^t R_t}$$

(Where t = the time at which a payment is made and R_t = amount of payment at time t)

The duration for the two-bond portfolio is:

$$\begin{aligned}
\bar{d} &= \frac{[90v + 2(90v^2) + ... + 10(90v^{10})] + 10(1{,}000v^{10}) + 13(1{,}000v^{13})}{[90v + (90v^2) + ... + (90v^{10})] + (1{,}000v^{10}) + (1{,}000v^{13})} \\
&= \frac{10{,}000v^{10} + 13{,}000v^{13} + 90(Ia)_{\overline{10}|}}{1{,}000v^{10} + 1{,}000v^{13} + 90a_{\overline{10}|}} \\
&= 8.47
\end{aligned}$$

Note that this is the MacCaulay duration. The modified duration can be determined as follows:

$$\text{Modified duration} = \bar{d}/(1+i)$$
$$= 8.47/1.09$$
$$= 7.77$$

Answer is A.

Question 12

Since 40% of the bond will be called after 5 years, the remaining 60% (or $600) will be paid upon maturity after 10 years. The equation of value is:

Price = Present value of future payments

$$620 = 320v^5 + 600v^{10}$$

$$0 = 600v^{10} + 320v^5 - 620$$

This can be written as:

$$0 = 600x^2 + 320x - 620 \quad (\text{where } x = v^5)$$
$$= 30x^2 + 16x - 31$$

Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, a = 30, b = 16, and c = -31.

Substituting into the quadratic formula:

$$x = \frac{-16 \pm \sqrt{(16)^2 - (4)(30)(-31)}}{(2)(30)} = .784259$$

Since x = .784259 and x = v^5,

$v^5 = .784259$
$v = .952559$
$i = 4.98\%$

Answer is B.

Test taking note: The correct range can be determined by using the interest rate at the endpoint of each range to determine which range must contain the correct yield rate.

Using $i = 4.95\%$, $320v^5 + 600v^{10} = 621.43$

Using $i = 5.05\%$, $320v^5 + 600v^{10} = 616.73$

Since the price of the bond is \$620, the yield rate must be between 4.95% and 5.05%.

Question 13

Step I: Determine the probability of death and withdrawal.

Using standard approximations:

$$q'^{(d)} = q^{(d)}/[1 - ½q^{(w)}]$$

$$.035 = q^{(d)}/[1 - ½(5q^{(d)})]$$

$$q^{(d)} = .032184$$

$$q^{(w)} = 5q^{(d)} = .160920$$

Step II: Determine the absolute rate of withdrawal.

$$q'^{(w)} = q^{(w)}/[1 - ½q^{(d)}] = .160920/[1 - ½(.032184)] = .163549$$

Answer is B.

Question 14

The probability that at least one of Smith and Brown will die during the next year is equal to one minus the probability that they both live. This is:

$$1 - (p_x)^2$$

The probability that both die during the next year is equal to:

$$q_x^2 = (1 - p_x)^2$$

Based upon the given information,

$$1 - (p_x)^2 = 20(1 - p_x)^2$$

Factoring,

$$(1 - p_x)(1 + p_x) = 20(1 - p_x)^2$$

$$1 + p_x = 20(1 - p_x)$$

$p_x = .904762$

There are two ways that exactly one of Smith and Brown could die during the next year. Either Smith could die and Brown could live, or Smith could live and Brown could die. The probability is:

$(2)(p_x)(1 - p_x) = (2)(.904762)(1 - .904762) = .172336$

Alternatively, the probability can be determined as equal to 1 minus the probability that they both live or they both die. This is:

$1 - p_x^2 - (1 - p_x)^2 = 1 - (.904762)^2 - (1 - .904762)^2 = .172336$

Answer is D.

Question 15

Using the first and third equations,

$$\begin{aligned}
{}_{10}p_{25:35:45} &= {}_{10}p_{25} \times {}_{10}p_{35} \times {}_{10}p_{45} \\
&= l_{35}/l_{25} \times l_{45}/l_{35} \times l_{55}/l_{45} \\
&= l_{55}/l_{25} \\
&= l_{40}/l_{25} \times l_{55}/l_{40} \\
&= {}_{15}p_{25} \times {}_{15}p_{40} \\
&= .975\ {}_{15}p_{40}
\end{aligned}$$

$.770 = .975\ {}_{15}p_{40}$

${}_{15}p_{40} = .789744$

Using the second equation and the above result,

$({}_5p_{45:50})({}_5q_{40}) = {}_{10}p_{45}\ (1 - {}_5p_{40}) = {}_{10}p_{45} - {}_{15}p_{40} = {}_{10}p_{45} - .789744$

$.029 = {}_{10}p_{45} - .789744$

${}_{10}p_{45} = .818744$

Using each of the above two results,

${}_{15}p_{40} = {}_5p_{40} \times {}_{10}p_{45} = {}_5p_{40} \times .818744$

$.789744 = {}_5p_{40} \times .818744$

${}_5p_{40} = .964580$

Finally, the question can be solved:

${}_{20}p_{25} = {}_{15}p_{25} \times {}_5p_{40} = .975 \times .964580 = .940466$

Answer is D.

Question 16

Note that ${}_{n|}q_x = {}_np_x - {}_{n+1}p_x$.

So,

$p_{105} - {}_2p_{105} = .2667$
${}_2p_{105} - {}_3p_{105} = .1000$
${}_3p_{105} - {}_4p_{105} = .0267$
${}_4p_{105} - {}_5p_{105} = .0067$
${}_5p_{105} = 0$

Substituting in successive equations,

${}_4p_{105} = .0067$
${}_3p_{105} = .0334$
${}_2p_{105} = .1334$
$p_{105} = .4001$

The present value can be written (using first principles):

$$1{,}000a_{105} = 1{,}000vp_{105} + 1{,}000v^2{}_2p_{105} + 1{,}000v^3{}_3p_{105} + 1{,}000v^4{}_4p_{105} = 516.27$$

Answer is A.

Question 17

Note that the formula for the curtate life expectancy at successive ages is:

$e_x = p_x(1 + e_{x+1})$

Also, the curtate life expectancy is approximated by the complete life expectancy less ½ (e - ½).

So,

$e_{50} = p_{50}(1 + e_{51}) \quad \Rightarrow \quad (23.2 - .5) = p_{50}(1 + (22.4 - .5)) \quad \Rightarrow \quad p_{50} = .991266$

$e_{51} = p_{51}(1 + e_{52}) \quad \Rightarrow \quad (22.4 - .5) = p_{51}(1 + (21.7 - .5)) \quad \Rightarrow \quad p_{51} = .986486$

$e_{52} = p_{52}(1 + e_{53}) \quad \Rightarrow \quad (21.7 - .5) = p_{52}(1 + (20.9 - .5)) \quad \Rightarrow \quad p_{52} = .990654$

The question can be solved:

$${}_3q_{50} = 1 - {}_3p_{50} = 1 - (.991266)(.986486)(.990654) = .031269$$

Answer is E.

Question 18

The annuity to Smith consists of the following:

\$200 payable as long as Smith is alive.
\$100 payable as long as Smith is alive after the death of Brown.
\$400 payable as long as Smith is alive after the death of Green.
\$500 payable as long as Smith is alive after the death of both Brown and Green.

The present value is:

$$\begin{aligned} PV &= 200a_x + 100a_{y|x} + 400a_{z|x} + 500\,a_{\overline{yz}|x} \\ &= 200a_x + 100(a_x - a_{xy}) + 400(a_x - a_{xz}) + 500(a_x - a_{xy} - a_{xz} + a_{xyz}) \\ &= 1{,}200a_x - 600a_{xy} - 900a_{xz} + 500\ _{xyz} \\ &= 5{,}200 \end{aligned}$$

Answer is D.

Question 19

Key concept: Since the expense load is equal to 7% of the net premium, the gross premium is equal to the present value of the benefits times 1.07.

For Policy A, the benefit is paid only if at least one of Smith, Brown and Green dies during the year. That probability is equal to 1 minus the probability that they all live. Using the first initial of each to represent their respective ages, the present value of the benefit is:

$$PVB = 100{,}000v(1 - p_{SBG})$$

The equation reflecting the gross premium is:

$$8{,}000 = 100{,}000v(1 - p_{SBG}) \times 1.07$$

Solving,

$$p_{SBG} = .92$$

For Policy B, the benefit is paid only if at least one of Smith or Green dies during the year. That probability is equal to 1 minus the probability that they both live. The present value of the benefit is:

$$PVB = 100{,}000v(1 - p_{SG})$$

The equation reflecting the gross premium is:

$$5{,}000 = 100{,}000v(1 - p_{SG}) \times 1.07$$

Solving,

$$p_{SG} = .95$$

Using the results from above,

$$p_{SBG} = p_{SG} \times p_B$$

$$.92 = .95 p_B$$

$$p_B = .968421$$

Answer is C.

Question 20

The equation of value for annuity A (using the first initial of each of Smith and Brown to represent their respective ages) is:

$$34{,}000 = 2{,}000\, a_{\overline{SB}} = 2{,}000 \times (a_S + a_B - a_{SB})$$

The equation of value for Annuity B is:

$$\begin{aligned} 50{,}000 &= 4{,}000 a_{SB} + (2{,}000)(a_{B|S} + a_{S|B}) = 4{,}000 a_{SB} + (2{,}000)(a_S + a_B - 2a_{SB}) \\ &= 2{,}000 \times (a_S + a_B) \end{aligned}$$

$$a_S + a_B = 25$$

Substituting into the equation for Annuity A,

$$34{,}000 = 50{,}000 - 2{,}000 a_{SB}$$

$$a_{SB} = 8$$

The present value of annuity C is:

$$\begin{aligned} PV &= 5{,}000 a_{SB} + (3{,}000)(a_{B|S} + a_{S|B}) \\ &= 5{,}000 a_{SB} + (3{,}000)(a_S + a_B - 2a_{SB}) \\ &= (3{,}000)(a_S + a_B) - 1{,}000 a_{SB} \\ &= (3{,}000)(25) - (1{,}000)(8) \\ &= 67{,}000 \end{aligned}$$

Answer is B.

Alternative Solution:

Using the equation of value for annuity A:

$$34{,}000 = 2{,}000\, a_{\overline{SB}} \qquad \Rightarrow \qquad a_{\overline{SB}} = 17$$

Annuity B can be thought of as an annuity of \$2,000 per year payable as long as either Smith or Brown is alive, plus an annuity of \$2,000 payable as long as both are alive. The equation of value is:

$$50{,}000 = 2{,}000\, a_{\overline{SB}} + 2{,}000 a_{SB} = 34{,}000 + 2{,}000 a_{SB}$$

$$a_{SB} = 8$$

Finally, annuity C can be thought of as an annuity of \$3,000 per year payable as long as either Smith or Brown is alive, plus an annuity of \$2,000 payable as long as both are alive. The equation of value is:

$$PV = 3{,}000\, a_{\overline{SB}} + 2{,}000 a_{SB} = (3{,}000)(17) + (2{,}000)(8) = 67{,}000$$

Answer is B.

Question 21

The initial employee contribution for Smith is:

$$\text{EE contribution} = 35{,}000 \times .05 = 1{,}750$$

The accumulated employee contributions as of 1/1/98 is:

$$\begin{aligned}\text{Accum EE contribution} &= 1{,}750 \times [1.07^9 + (1.04)(1.07^8) + \dots + (1.04^8)(1.07) + 1.04^9] \\ &= 1{,}750 \times 1.07^9 \times [1 + (1.04/1.07) + \dots + (1.04/1.07)^9] \\ &= 1{,}750 \times 1.07^9 \times \ddot{a}_{\overline{10|}j} \quad (\text{where } j = 1.07/1.04 - 1 = .028846) \\ &= 1{,}750 \times 1.838459 \times 8.828179 \\ &= 28{,}403\end{aligned}$$

Since the valuation interest rate is the same as the interest credited on the employee contributions, there is only a mortality discount to be applied in the determination of the present value of the pre-retirement death benefit.

It is necessary to solve for ${}_{20}q_{45}$.

$$D_{65}/D_{45} = v^{20}\,{}_{20}p_{45} \quad \Rightarrow \quad {}_{20}p_{45} = .820895 \quad \Rightarrow \quad {}_{20}q_{45} = .179105$$

Now the present value of the pre-retirement death benefit can be determined.

$$PV = 28{,}403\ {}_{20}q_{45} = 28{,}403 \times .179105 = 5{,}087$$

Answer is B.

Question 22

Key concept: The last age at which a benefit will be paid is age 99, since at age 100 the value of l_x is zero.

Using first principles:

$$
\begin{aligned}
a_{70} &= vp_{70} + v^2{}_2p_{70} + \ldots + v^{29}{}_{29}p_{70} \\
&= (29/30)v + (28/30)v^2 + \ldots + (1/30)v^{29} \\
&= (1/30) \times (29v + 28v^2 + \ldots + v^{29}) \\
&= (1/30)\ (Da)_{\overline{29|}} \\
&= (1/30)[(29 - a_{\overline{29|}})/i] \\
&= (1/30)[(29 - 13.590721)/.06] \\
&= 8.560711
\end{aligned}
$$

Answer is B.

Question 23

The annual contribution to the savings account is:

$$\text{Contribution} = 100{,}000/\ddot{s}_{\overline{5|}} = 100{,}000/6.153291 = 16{,}251$$

The present value of the premiums must be equal to the present value of the future benefits. The death benefit in any given year is equal to 100,000 less the accumulated contributions as of the end of the year.

The equation of value (where P is the annual premium) is:

$$
\begin{aligned}
P\ddot{a}_{40:\overline{5|}} &= (100{,}000 - 16{,}251\,\ddot{s}_{\overline{1|}})vq_{40} + (100{,}000 - 16{,}251\,\ddot{s}_{\overline{2|}})v^2{}_{1|}q_{40} + \\
&\qquad \ldots + (100{,}000 - 16{,}251\,\ddot{s}_{\overline{5|}})v^5{}_{4|}q_{40} \\
&= 100{,}000\,A^1_{40:\overline{5|}} - (16{,}251)(\ddot{s}_{\overline{1|}}\,vq_{40} + \ddot{s}_{\overline{2|}}\,v^2{}_{1|}q_{40} + \ldots + \ddot{s}_{\overline{5|}}\,v^5{}_{4|}q_{40}) \\
&= 100{,}000\,A^1_{40:\overline{5|}} \\
&\qquad - (16{,}251)[((1.07 - 1)/d)vq_{40} + ((1.07^2 - 1)/d)v^2{}_{1|}q_{40} + \ldots + ((1.07^5 - 1)/d)v^5{}_{4|}q_{40}] \\
&= 100{,}000\,A^1_{40:\overline{5|}} \\
&\qquad - (16{,}251/d)[(1.07 - 1)vq_{40} + (1.07^2 - 1)v^2{}_{1|}q_{40} + \ldots + (1.07^5 - 1)v^5{}_{4|}q_{40}] \\
&= 100{,}000\,A^1_{40:\overline{5|}} \\
&\qquad - (16{,}251/d)[(q_{40} + {}_{1|}q_{40} + \ldots + {}_{4|}q_{40}) - (vq_{40} + v^2{}_{1|}q_{40} + \ldots + v^5{}_{4|}q_{40})] \\
&= 100{,}000\,A^1_{40:\overline{5|}} - (16{,}251/d)[{}_5q_{40} - A^1_{40:\overline{5|}}]
\end{aligned}
$$

It is necessary to determine the values for ${}_5q_{40}$, $\ddot{a}_{40:\overline{5|}}$, and $A^{1}_{40:\overline{5|}}$.

$$D_{45}/D_{40} = (N_{45} - N_{46})/(N_{40} - N_{41}) = v^5\,{}_5p_{40}$$

$\Rightarrow$ $5{,}690{,}850 - 5{,}245{,}842)/(8{,}452{,}729 - 7{,}820{,}455) = (1/1.07)^5\,{}_5p_{40}$

$\Rightarrow$ ${}_5p_{40} = .987146$

$\Rightarrow$ ${}_5q_{40} = 1 - .987146 = .012854$

$$\ddot{a}_{40:\overline{5|}} = (N_{40} - N_{45})/D_{40} = (N_{40} - N_{45})/(N_{40} - N_{41}) = 4.368168$$

$$A^{1}_{40:\overline{5|}} = (M_{40} - M_{45})/D_{40} = ([vN_{40} - N_{41}] - [vN_{45} - N_{46}])/(N_{40} - N_{41}) = .010412$$

Substituting into the equation of value, above:

$$P\,\ddot{a}_{40:\overline{5|}} = 100{,}000\,A^{1}_{40:\overline{5|}} - (16{,}251/d)[{}_5q_{40} - A^{1}_{40:\overline{5|}}]$$

$\Rightarrow$ $4.368168P = (100{,}000)(.010412) - (16{,}251/[.07/1.07])[.012854 - .010412]$

$\Rightarrow$ $P = 99.49$

Answer is A.

Question 24

Step I: Determine the annual retirement benefit.

The salary must be projected to the year 2000.

Final salary = $40{,}000 \times 1.04^2 = 43{,}264$

Annual retirement benefit = $43{,}264 \times .5 = 21{,}632$

Step II: Determine the present value of the retirement benefit.

$$\begin{aligned} PV_{\text{Retben}} &= 21{,}632\,\ddot{a}^{(12)}_{65}\; v^3\,{}_3p_{62} \\ &= 21{,}632 \times 8.74 \times .816298 \times (1 - .0170)(1 - .0187)(1 - .0205) \\ &= 145{,}820 \end{aligned}$$

Step III: Determine the present value of the pre-retirement death benefit.

$$PV_{\text{Deathben}} = (80{,}000)vq_{62} + (80{,}000)(1.04)v^2\,{}_{1|}q_{62} + (80{,}000)(1.04^2)v^3\,{}_{2|}q_{62} = 4{,}004$$

Step IV: Determine the total present value.

$$PV_{\text{Tot}} = PV_{\text{Retben}} + PV_{\text{Deathben}} = 145{,}820 + 4{,}004 = 149{,}824$$

Answer is C.

Question 25

It is first necessary to determine the values of p_{64}, p_{65}, and p_{66}.

$$D_{65}/D_{64} = (N_{65} - N_{66})/(N_{64} - N_{65}) = vp_{64} \quad \Rightarrow \quad p_{64} = .979481$$
$$D_{66}/D_{65} = (N_{66} - N_{67})/(N_{65} - N_{66}) = vp_{65} \quad \Rightarrow \quad p_{65} = .977436$$
$$D_{67}/D_{66} = (N_{67} - N_{68})/(N_{66} - N_{67}) = vp_{66} \quad \Rightarrow \quad p_{66} = .975139$$

The annuity value can now be determined:

$$\begin{aligned} a_{64:65:\overline{2}|} &= vp_{64:65} + v^2{}_2p_{64:65} \\ &= v(.979481)(.977436) + v^2(.979481)(.977436)(.977436)(.975139) \\ &= 1.691772 \end{aligned}$$

Answer is C.

Solutions to EA-1(A) Examination
Spring, 1999

Question 1

Step I: Calculate the time-weighted rate of return.

$$(1+t) = \frac{\text{Balance on 3/31/99*}}{\text{Balance on 1/1/99}} \times \frac{\text{Balance on 9/30/99*}}{\text{Balance on 4/1/99}} \times \frac{\text{Balance on 1/1/2000}}{\text{Balance on 10/1/99}}$$

$$= \frac{90{,}000 + 20{,}000}{100{,}000} \times \frac{185{,}000 - 75{,}000}{90{,}000} \times \frac{180{,}000}{185{,}000}$$

$$= 1.3081$$

$$\Rightarrow \quad t = 30.81\%$$

* Balance before contribution or benefit payment

Step II: Calculate the dollar-weighted rate of return.

Ending Balance = (Beginning Balance)$(1 + d)$ + (Transaction)(1+fractional interest)

$180{,}000 = (100{,}000)(1 + d) - (20{,}000)(1 + \frac{3}{4}d) + (75{,}000)(1 + \frac{1}{4}d)$

Solving for d, $d = 24.10\%$

Note: The equation of value could have been written using compound interest. However, the result would not have been significantly different.

Step III: Calculate the dollar weighted rate of return assuming the contributions and benefit payments occur at mid-year (7/1/99).

Ending Balance = (Beginning Balance)$(1 + m)$ + (Transaction)$(1 + \frac{1}{2}m)$

$180{,}000 = (100{,}000)(1 + m) + (55{,}000)(1 + \frac{1}{2}m)$

Solving for m, $m = 19.61\%$

Therefore, $t > d > m$.

Answer is B.

Question 2

Convert the annual rate of interest, compounded semiannually, to an annual effective rate of interest.

$$i = (1 + i^{(2)}/2)^2 - 1 = (1.04)^2 - 1 = .0816$$

The payment stream can be viewed as annual payments of \$2,000, offset by annual payments of \$500 for the first 20 years, and further offset by annual payments of \$500 for the first 10 years, with a total of 25 payments. The present value of this stream of payments is:

$$\begin{aligned}\text{Present value} &= 2{,}000\, a_{\overline{25|}} - 500\, a_{\overline{20|}} - 500\, a_{\overline{10|}} \\ &= 21{,}061 - 4{,}851 - 3{,}331 \\ &= 12{,}879\end{aligned}$$

Answer is B.

Question 3

The monthly effective rate of interest is:

$$i^{(12)}/12 = .08/12 = .006667$$

The quarterly effective rate of interest is:

$$i^{(4)}/4 = (1 + i^{(12)}/12)^3 - 1 = .020134$$

The payment stream of annuity A can be thought of as monthly payments of \$1,000 for 12 months, reduced by monthly payments of \$500 for the first 3 months.

The payment stream of annuity B can be thought of as quarterly payments of \$2P for 4 quarters, reduced by quarterly payments of \$P for the first 2 quarters.

The present value of the two annuities can be set equal to each other.

$$1{,}000\, a_{\overline{12|}.006667} - 500\, a_{\overline{3|}.006667} = 2P\, a_{\overline{4|}.020134} - P\, a_{\overline{2|}.020134}$$

$$11{,}496 - 1{,}480 = 7.612982P - 1.941180P$$

$$\Rightarrow \qquad P = 1{,}766$$

Answer is A.

Question 4

The monthly effective rate of interest is:

$i^{(12)}/12 = .108/12 = .009$

The formula for duration is:

$$\bar{d} = \frac{\sum_{t=1}^{n} t v^t R_t}{\sum_{t=1}^{n} v^t R_t}$$

(Where t = the time at which a payment is made and R_t = amount of payment at time t)

The duration for the loan is:

$$\begin{aligned}
\bar{d} &= (v + 2v^2 + \ldots + 360v^{360})/(v + v^2 + \ldots + v^{360}) \\
&= (Ia)_{\overline{360|}} / a_{\overline{360|}} \\
&= [(\ddot{a}_{\overline{360|}} - 360v^{360})/.009]/ a_{\overline{360|}} \\
&= 97.21
\end{aligned}$$

Note that this is the MacCaulay duration. The modified duration can be determined as follows:

$$\begin{aligned}
\text{Modified duration} &= \bar{d}/(1 + i^{(12)}/12) \\
&= 97.21/1.009 \\
&= 96.34
\end{aligned}$$

Answer is C.

Question 5

The contribution, y, must be determined for the 3rd quarter. The fund lost 2.26% for that quarter, which means that the ending value (before the contribution was deposited) was equal to the 97.74% of the beginning balance (after the benefit payment was made). Therefore,

$$.9774 = (1{,}937{,}033 - y)/(1{,}907{,}549 - 27{,}994) \quad \Rightarrow \quad y = 99{,}956$$

The time-weighted rate of return is equal to the product of the ratios of the beginning balances (after the benefit payment has been made) to the ending balances (before the contribution has been made) for each period during which a transaction has occurred. This is reflected by the following equation.

$$\begin{aligned}
1.0879 = {} & ([1{,}775{,}626 - 30{,}000]/[1{,}750{,}000 - 22{,}000]) \\
& \times ([1{,}907{,}549 - 85{,}000]/[1{,}775{,}626 - 43{,}000]) \\
& \times ([1{,}937{,}033 - 99{,}956]/[1{,}907{,}549 - 27{,}994]) \times ([z - 27{,}617]/[1{,}937{,}033 - 39{,}228])
\end{aligned}$$

$\Rightarrow \quad z = 2{,}015{,}480$

The dollar-weighted rate of return, i, can now be determined.

$$\text{Ending Balance} = (\text{Beginning Balance})(1 + i) + (\text{Transaction})(1+\text{fractional interest})$$
$$\begin{aligned}2{,}015{,}480 = {} & (1{,}750{,}000)(1 + i) - (22{,}000)(1 + i) + (30{,}000)(1 + \tfrac{3}{4}i) - (43{,}000)(1 + \tfrac{3}{4}i) \\ & + (85{,}000)(1 + \tfrac{1}{2}i) - (27{,}994)(1 + \tfrac{1}{2}i) + (99{,}956)(1 + \tfrac{1}{4}i) \\ & - (39{,}228)(1 + \tfrac{1}{4}i) + 27{,}617\end{aligned}$$

$$\Rightarrow \qquad i = 8.80\%$$

Answer is D.

Alternative solution: First solve for y, as above. The quarterly return for the 4th quarter can be determined.

$$1.0879 = 1.0102 \times 1.0519 \times .9774 \times (1 + j\%) \qquad \Rightarrow \qquad j\% = 4.7454\%$$

The ending balance is:

$$\text{Balance}_{12/31/99} = (1{,}937{,}033 - 39{,}228)(1.047454) + 27{,}617 = 2{,}015{,}548$$

Note that the difference in the ending balance is due to interest rate rounding.

Solving for the dollar-weighted rate of return as above, the rate is 8.81%. Again, the difference is due to rounding.

Question 6

Step I: Determine the outstanding balance of the loan immediately prior to the 53rd payment.

The monthly interest rate of the original loan is:

$$i^{(12)}/12 = (1 + i)^{1/12} - 1 = (1.07)^{1/12} - 1 = .005654$$

The outstanding balance after the 52nd payment is:

$$\begin{aligned}\text{OutBal}_{5/1/2003} &= 10{,}000 \times (a_{\overline{68|}} / a_{\overline{120|}}) \\ &= 6{,}477\end{aligned}$$

The outstanding balance immediately prior to the 53rd payment is:

$$\text{OutBal}_{6/1/2003} = 6{,}477 \times 1.005654 = 6{,}514$$

Step II: Determine \$P.

The annual effective interest rate for the savings account is:

$$i = (1 + i^{(12)}/12)^{12} - 1 = (1 + .09/12)^{12} - 1 = .093807$$

Also note that $i^{(12)}/12 = .09/12 = .0075$.

The accumulated value of the savings account as of June 1, 2003 is:

$$\text{Savings}_{6/1/2003} = P \times s_{\overline{4|}.093807} \times 1.0075^5 = 4.773930P$$

The accumulated savings account is equal to the outstanding loan balance plus a 10% prepayment penalty.

$$4.773930P = 6{,}514 \times 1.1 \qquad \Rightarrow \qquad P = 1{,}501$$

Answer is D.

Question 7

Under the original terms of the mortgage, the monthly payment is:

$$\text{Payment} = 100{,}000 / a_{\overline{360|}.01} = 1{,}028.61$$

The total number of repayments, N, must be determined assuming that an additional payment of \$3,000 is paid on 12/31/98. The equation of balance is:

$$100{,}000 = 1{,}028.61\, a_{\overline{N|}.01} + 3{,}000 v^{60}_{.01}$$

$$a_{\overline{N|}.01} = 95.613159$$

$$(1 - v^N)/.01 = 95.613159$$

$$v^N = .043868$$

$$\ln(v^N) = \ln(.043868)$$

$$N = 314.2$$

Therefore, the final 315th payment will be a smaller payment than 1,028.61. The equation of balance reflecting the final payment, F, is:

$$100{,}000 = 1{,}028.61\, a_{\overline{314|}.01} + 3{,}000 v^{60}_{.01} + F v^{315}_{.01} \qquad \Rightarrow \qquad F = 223.83$$

The total interest paid is equal to the total payments made less the original loan balance.

Total payments = (1,028.61 × 314) + 3,000 + 223.83 = 326,207.37
Total interest paid = 326,207.37 - 100,000.00 = 226,207.37

Answer is B.

Question 8

The equation of value for the perpetuity is:

$$P = 2X\,a_{\overline{\infty|}}$$

$$P/2 = X\,a_{\overline{\infty|}}$$

The equation of value for the 10-year annuity certain is:

$$P/2 = X\,s_{\overline{10|}}$$

Setting the equations equal to each other:

$$X\,s_{\overline{10|}} = X\,a_{\overline{\infty|}}$$

$$s_{\overline{10|}} = a_{\overline{\infty|}}$$

$$[(1 + i)^{10} - 1]/i = 1/i$$

$$(1 + i)^{10} = 2$$

$$i = .0718, \text{ or } 7.18\%$$

Answer is A.

Question 9

$i^{(12)}/12 = .08/12 = .006667$ and $i^{(4)}/4 = (1 + i^{(12)}/12)^3 - 1 = 1.006667^3 - 1 = .020135$

The fund balance is equal to the accumulated initial balance plus the accumulated deposits, less the accumulated withdrawals.

$$\begin{aligned}\text{Balance}_{12/31/2010} &= [12{,}000 \times 1.006667^{144}] + [100\,s_{\overline{60|}.006667} \times 1.006667^{84}] \\ &\quad - [1{,}000\,\ddot{s}_{\overline{20|}.020135}] \\ &= 31{,}242 + 12{,}840 - 24{,}820 \\ &= 19{,}262\end{aligned}$$

Answer is E.

Question 10

Step I: Determine the present value of one year's payments.

The quarterly effective rate of interest can be calculated since the payments are made quarterly.

$$i^{(4)}/4 = 1.1^{1/4} - 1 = .024114$$

The present value of one year's payments is:

$$100\,(Ia)_{\overline{4|}} = 100 \times [(\ddot{a}_{\overline{4|}} - 4v^4)/i] = 931.28$$

Step II: Determine the purchase price (present value) of the perpetuity.

The perpetuity can be thought of as annual payments of \$931.28 made at the beginning of each year.

$$\text{Price} = 931.28\,\ddot{a}_{\overline{\infty}|.1} = 931.28 \times 1/d = 10{,}244$$

Answer is B.

Question 11

The yield rate every six months is 5.5%/2, or 2.75%, and the coupon reinvestment rate every six months is 5%/2, or 2.5%.

Each coupon is 3% of the face amount of the bond, or \$3,000.

Since the coupons are reinvested at a different interest rate than the yield rate, the purchase price can be found by looking at accumulated values. The accumulated equation of value at the redemption date of 1/1/2009 is:

$$\text{Price} \times 1.0275^{20} = 3{,}000\,s_{\overline{20}|.025} + 100{,}000 \qquad \Rightarrow \qquad \text{Price} = 102{,}669$$

Answer is D.

Question 12

Recall the identity $A_x = 1 - d\ddot{a}_x$. So,

$$A_{45} = 1 - d\ddot{a}_{45}$$

The present value of the net annual premiums is equal to the present value of the benefits. So,

$$45\ddot{a}_{45} = 1{,}000A_{45} = 1{,}000 \times (1 - d\ddot{a}_{45})$$

$$\ddot{a}_{45} = 9.842154$$

Note that $d = i/(1 + i) = .06/1.06 = .056604$

$$A_{45} = 1 - d\ddot{a}_{45} = 1 - (.056604)(9.842154) = .442895$$

The net single premium is equal to the present value of the death benefits. So,

$$Y = 1{,}000A_{45} = 1{,}000 \times .442895 = 442.90$$

Answer is B.

Shortcut: The net single premium is equal to the present value of the net annual premium.

$$Y = 45\ddot{a}_{45} = 45 \times 9.842154 = 442.90$$

Question 13

Each term of the annuity can be valued separately.

If each beneficiary is still alive, then each receives \$400 per month. The value of this (PV_1) is:

$$PV_1 = 400 \times 12 \times 3 \text{ beneficiaries} \times \ddot{a}^{(12)}_{yyy} = 90{,}115$$

If exactly one beneficiary has died, then each remaining beneficiary receives \$450 per month. The value of this (PV_2) is:

$$PV_2 = 450 \times 12 \times 2 \text{ beneficiaries} \times 3 \text{ possible deaths} \times (\ddot{a}^{(12)}_{yy} - \ddot{a}^{(12)}_{yyy}) = 35{,}510$$

Note that the reversionary annuity $\ddot{a}_{y|yy} = \ddot{a}_{yy} - \ddot{a}_{yyy}$.

If only one beneficiary is surviving, that beneficiary receives \$X per month. The value of this (PV_3) is:

$$PV_3 = X \times 12 \times 3 \text{ beneficiaries} \times (\ddot{a}^{(12)}_{y} - 2\ddot{a}^{(12)}_{yy} + \ddot{a}^{(12)}_{yyy}) = 26.784X$$

Note that the reversionary annuity $\ddot{a}_{yy|y} = \ddot{a}_{y} - 2\ddot{a}_{yy} + \ddot{a}_{yyy}$.

The sum of the three present values must be equal to 140,000.

$$140{,}000 = 90{,}115 + 35{,}510 + 26.784X \quad \Rightarrow \quad X = 536.70$$

Answer is C.

Question 14

Key concept: The probability of receiving a death benefit or a retirement benefit is equal to one minus the probability that neither benefit is received. Since the participant was hired at age 58, there will be no death benefit or retirement benefit payable if termination or death occurs before age 63.

It is given that $q'^{(d)}_x = .02$ and $q'^{(w)}_x = .04$ at each age x.

The probability of death or termination before attainment of age 61 is:

$$1 - (p'^{(d)}_{60})(p'^{(w)}_{60}) = 1 - (.98)(.96) = .0592$$

The probability of death or termination between age 61 and 62 is:

$$(p'^{(d)}_{60})(p'^{(w)}_{60})(1 - (p'^{(d)}_{61})(p'^{(w)}_{61})) = (.98)(.96)(1 - (.98)(.96)) = .0557$$

The probability of death or termination between age 62 and 63 is:

$$({}_2p'^{(d)}_{60})({}_2p'^{(w)}_{60})(1 - (p'^{(d)}_{62})(p'^{(w)}_{62})) = (.98^2)(.96^2)(1 - (.98)(.96)) = .0524$$

The total probability that death or termination occurs before age 63 is:

$.0592 + .0557 + .0524 = .1673$

The probability that a death or retirement benefit will be paid is:

$1 - .1673 = .8327$

Answer is D.

Alternative solution: The employee will receive a benefit (either a retirement benefit or a death benefit) if he survives for 3 more years (to age 63). This probability is:

$${}_3p_{60}^{(T)} = (p_{60}^{\prime(d)})(p_{60}^{\prime(w)})(p_{61}^{\prime(d)})(p_{61}^{\prime(w)})(p_{62}^{\prime(d)})(p_{62}^{\prime(w)}) = [(.98)(.96)]^3 = .8327$$

Question 15

Green must either die before 2004 or after 2004. This is the probability that Green dies within 5 years, plus the probability that Green lives at least 6 years. That probability is:

${}_5q_{63} + {}_6p_{63} = (l_{63} - l_{68})/l_{63} + l_{69}/l_{63} = (944 - 849)/944 + 830/944 = .97987$

The probability that either Smith or Brown die in 2004 is equal to one minus the probability that neither dies in 2004. That probability is:

$$\begin{aligned} 1 - ({}_5q_{60} + {}_6p_{60})({}_5q_{65} + {}_6p_{65}) &= 1 - ([(l_{60} - l_{65})/l_{60} + l_{66}/l_{60}][(l_{65} - l_{70})/l_{65} + l_{71}/l_{65}] \\ &= 1 - ([1{,}000 - 906)/1{,}000 + 887/1{,}000][(906 - 811)/906 + 792/906] \\ &= .03957 \end{aligned}$$

The total probability is: $(.97987)(.03957) = .03877$

Answer is D.

Question 16

Recall the formula for annuities due at successive ages:

$\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$

So,

$\ddot{a}_{50} = 1 + vp_{50}\ddot{a}_{51}$

$\Rightarrow$ $12.07872 = 1 + (1/1.07)(11.92117)p_{50}$

$\Rightarrow$ $p_{50} = .994385$

And,

$\ddot{a}_{51} = 1 + vp_{51}\ddot{a}_{52}$

$\Rightarrow$ $11.92117 = 1 + (1/1.07)(11.75854)p_{51}$

$\Rightarrow$ $p_{51} = .993801$

The probability that an annuitant age 50 will die before age 52 is:

$$
\begin{aligned}
{}_2q_{50} &= 1 - {}_2p_{50} \\
&= 1 - (.994385)(.993801) \\
&= .0118
\end{aligned}
$$

Answer is C.

Question 17

The equation of value for Contract 1 is:

$$6{,}000 = 1{,}000\, a_{30:\overline{15|}}$$

$$\Rightarrow \quad a_{30:\overline{15|}} = 6$$

The equation of value for Contract 2 is:

$$75{,}000 = 5{,}000\, \ddot{a}_{30:\overline{16|}} + 1{,}000\,(Ia)_{30:\overline{15|}}$$

$$\Rightarrow \quad 75{,}000 = 5{,}000 + 5{,}000\, a_{30:\overline{15|}} + 1{,}000\,(Ia)_{30:\overline{15|}}$$

$$\Rightarrow \quad 75{,}000 = 5{,}000 + (5{,}000)(6) + 1{,}000\,(Ia)_{30:\overline{15|}}$$

$$\Rightarrow \quad (Ia)_{30:\overline{15|}} = 40$$

The equation of value for Contract 3 is:

$$
\begin{aligned}
P &= 500\,(Da)_{30:\overline{15|}} \\
&= 8{,}000\, a_{30:\overline{15|}} - 500\,(Ia)_{30:\overline{15|}} \\
&= (8{,}000)(6) - (500)(40) \\
&= 28{,}000
\end{aligned}
$$

Answer is D.

Question 18

The present value of the normal retirement benefit is equal to the present value of the benefit under the terms of option A.

$$7{,}200\ddot{a}_{65}\,({}_{20}E_{45}) = (X)(\ddot{a}_{\overline{15|}}\ {}_{10}E_{45} + \ddot{a}_{70}\,({}_{25}E_{45}))$$

$$\Rightarrow \quad (7{,}200)(9.1941)(.2122) = (X)((9.7455)(.4808) + (8.0605)(.1317))$$

$$\Rightarrow \quad X = 2{,}444$$

Answer is D.

Question 19

The first selected value given, $A_{\overset{\;\;1}{60:\overline{1|}}}$, is the present value of a payment after one year if the life age 60 survives to age 61.

This can be rewritten as:

$$1{,}000\,A_{\overset{\;\;1}{60:\overline{1|}}} = 1{,}000vp_{60} = 921$$

$\Rightarrow \quad p_{60} = .98547$

And,

$p'_{60} = .99547$, where p'_{60} is the probability of survival with inclusion of the survivorship improvement.

Recall that the recursive formula for insurances is:

$$A_x = vq_x + vp_xA_{x+1}$$

So,

$1{,}000A_{60} = (1{,}000)(vq_{60} + vp_{60}A_{61}) = 328$

$\Rightarrow \quad A_{61} = .34139$

Note that the value of A_{61} does not change due to the mortality improvement at age 60, so:

$1{,}000\,A'_{60} = (1{,}000)(v\,q'_{60} + v\,p'_{60}\,A_{61}) = 321.844$

$\Rightarrow \quad A'_{60} = .321844$

Recall that:

$A_x = 1 - d\ddot{a}_x$

$\Rightarrow \quad \ddot{a}_x = (1 - A_x)/d$

So,

$\ddot{a}_{60}' = (1 - A_{60}')/d = 10.366099$

The net single premium (or the present value of the annuity) is:

$$15{,}000\,\ddot{a}'_{60} = (15{,}000)(10.366099) = 155{,}491$$

Answer is D.

Question 20

Recall the approximation:

$$\mu_{x+½} \cong -\ln(p_x)$$
$$\Rightarrow \quad \mu_{[40]+½} \cong -\ln(p_{[40]})$$
$$\Rightarrow \quad \ln(1.04) \cong -\ln(p_{[40]})$$
$$\Rightarrow \quad 1.04 \cong e^{-\ln(p[40])}$$
$$\Rightarrow \quad 1.04 \cong 1/p_{[40]}$$
$$\Rightarrow \quad p_{[40]} \cong .961538$$

At this point, we will assume that this is the exact value of $p_{[40]}$ and dispense with the continued use of the approximation sign. Therefore,

$$q_{[40]} = 1 - p_{[40]} = .038462$$

And,

$$q_{[40]} = 0.6q_{40}$$
$$\Rightarrow \quad q_{40} = .038462/.6 = .064103 \text{ and } p_{40} = .935897$$

Recall that the recursive formula for an annuity immediate is:

$$a_x = vp_x + vp_x a_{x+1}$$

So,

$$a_{40} = vp_{40} + vp_{40}a_{41}$$
$$\Rightarrow \quad 16 = (.935897)(v + va_{41})$$
$$\Rightarrow \quad v + va_{41} = 17.095898$$

Finally,

$$\begin{aligned} a_{[40]} &= vp_{[40]} + vp_{[40]}a_{41} \\ &= (p_{[40]})(v + va_{41}) \\ &= (.961538)(17.095898) \\ &= 16.438 \end{aligned}$$

Answer is C.

Question 21

$${}_np_{xx} = ({}_np_x)^2 = .25$$
$$\Rightarrow \quad {}_np_x = .5$$

$${}_{n|}q_{xxx} = ({}_np_{xxx})(q_{x+n:x+n:x+n})$$

Finally,

$$\begin{aligned} {}_nq_x + {}_nq_{xx} - {}_{n|}q_{xxx} &= (1 - {}_np_x) + (1 - {}_np_{xx}) - ({}_np_{xxx})(1 - p_{x+n:x+n:x+n}) \\ &= (1 - .5) + (1 - .5^2) - (.5^3)(1 - .5^3) \\ &= 1.140625 \end{aligned}$$

Answer is B.

Question 22

The present value of the net annual premiums is equal to the present value of the death benefits.

$$1{,}548\ddot{a}_{45} = 100{,}000A_{45}$$
$\Rightarrow$ $1{,}548 = (100{,}000)(A_{45}/\ddot{a}_{45})$
$\Rightarrow$ $1{,}548 = (100{,}000)(M_{45}/N_{45})$
$\Rightarrow$ $1{,}548 = (100{,}000)(M_{45}/4{,}450)$
$\Rightarrow$ $M_{45} = 68.886$

Recall the formula:

$$M_x = vN_x - N_{x+1}$$

Using the formula,

$$M_{45} = vN_{45} - N_{46}$$
$\Rightarrow$ $68.886 = 4{,}450v - 4{,}208$
$\Rightarrow$ $v = .961098$

and
$$M_{65} = vN_{65} - N_{66}$$
$\Rightarrow$ $M_{65} = (.961098)(1{,}060) – 960 = 58.764$

Also,
$$D_{45} = N_{45} - N_{46} = 4{,}450 - 4{,}208 = 242$$
$\Rightarrow$ $D_{65} = N_{65} - N_{66} = 1{,}060 - 960 = 100$

Finally,
$$\begin{aligned}100{,}000\,{}_{20}V_{45} &= 100{,}000A_{65} - 1{,}548\ddot{a}_{65}\\ &= (100{,}000)(M_{65}/D_{65}) - (1{,}548)(N_{65}/D_{65})\\ &= (100{,}000)(58.764/100) - (1{,}548)(1{,}060/100)\\ &= 42{,}355\end{aligned}$$
Answer is D.

Question 23

Key concept: The last age at which a benefit will be paid is age 109, since at age 110 the value of l_x is zero.

Using first principles:

$$\begin{aligned}\ddot{a}_{65} &= 1 + vp_{65} + v^2{}_2p_{65} + \ldots + v^{44}{}_{44}p_{65}\\ &= 1 + v(l_{66}/l_{65}) + v^2(l_{67}/l_{65}) + \ldots + v^{44}(l_{109}/l_{65})\\ &= 1 + (44/45)v + (43/45)v^2 + \ldots + (1/45)v^{44}\\ &= 1 + (1/45) \times (44v + 43v^2 + \ldots + v^{44})\\ &= 1 + (1/45)(Da)_{44}\\ &= 1 + (1/45)[(44 - a_{44})/i]\\ &= 1 + (1/45)[(44 - 13.557908)/.07]\\ &= 10.664\end{aligned}$$

Answer is B.

Question 24

Using first principles, the equation of value (where P is the annual payment) is:

$$100{,}000 = P \times [v(l_{[65]+1}/l_{[65]}) + v^2(l_{[65]+2}/l_{[65]}) + v^3(l_{65+3}/l_{[65]}) + v^4(l_{66+3}/l_{[65]})]$$
$$= P \times 3.197833$$

$\Rightarrow$ $P = 31{,}271$

Answer is B.

Question 25

Using first principles, the equation of value is:

$$(1{,}000)(12\ddot{a}_{65}^{(12)}) = 12P\ddot{a}_{65:\overline{1|}}^{(12)} + (1.03)(12P\ddot{a}_{66:\overline{1|}}^{(12)}(D_{66}/D_{65})) + (1.03^2)(12P\ddot{a}_{67:\overline{1|}}^{(12)}(D_{67}/D_{65}))$$
$$+ (1.03^3)(12P\ddot{a}_{68}^{(12)}(D_{68}/D_{65}))$$

The value of each annuity can be solved:

$$\ddot{a}_{65:\overline{1|}}^{(12)} = [(N_{65} - N_{66}) - (11/24)(D_{65} - D_{66})]/D_{65} = .960348$$
$$\ddot{a}_{66:\overline{1|}}^{(12)} = [(N_{66} - N_{67}) - (11/24)(D_{66} - D_{67})]/D_{66} = .959373$$
$$\ddot{a}_{67:\overline{1|}}^{(12)} = [(N_{67} - N_{68}) - (11/24)(D_{67} - D_{68})]/D_{67} = .958354$$
$$\ddot{a}_{65}^{(12)} = (N_{65} - (11/24)(D_{65}))/D_{65} = 8.735780$$
$$\ddot{a}_{68}^{(12)} = (N_{68} - (11/24)(D_{68}))/D_{68} = 8.061125$$

Using these values in the original equation of value,

$$(1{,}000)(12)(8.735780) = (12P)(.960348) + (1.03)(12P)(.959373)(D_{66}/D_{65}))$$
$$+ (1.03^2)(12P)(.958354)(D_{67}/D_{65}))$$
$$+ (1.03^3)(12P)(8.061125)(D_{68}/D_{65}))$$

$\Rightarrow$ $104{,}829 = (P)(11.524176 + 10.831997 + 10.157184 + 80.003523)$

$\Rightarrow$ $P = 931.67$

Answer is B.

Alternative solution: This can be treated as a series of four annuities. The first annuity calls for monthly payments of \$P for life, beginning at age 65. The second annuity calls for a payment of \$.03P for life, beginning at age 66. The third annuity calls for a payment of \$$(1.03^2 - 1.03)$P for life, beginning at age 67. The fourth annuity calls for a payment of \$$(1.03^3 - 1.03^2)$P for life, beginning at age 68. The equation of value is:

$$(1{,}000)(12\ddot{a}_{65}^{(12)}) = 12P\ddot{a}_{65}^{(12)} + (.03)(12P\ddot{a}_{66}^{(12)}(D_{66}/D_{65}))$$
$$+ (1.03^2 - 1.03)(12P\ddot{a}_{67}^{(12)}(D_{67}/D_{65}))$$
$$+ (1.03^3 - 1.03^2)(12P\ddot{a}_{68}^{(12)}(D_{68}/D_{65}))$$

$\Rightarrow$ $P = 931.70$

Solutions to EA-1(A) Examination
Spring, 2000

Question 1

The equation of value for the perpetuity-due is:

$1{,}100 = 100\,\ddot{a}_{\overline{\infty}|i}$

The interest rate, *i*, can be determined.

$11 = \ddot{a}_{\overline{\infty}|i} = 1/d \Rightarrow \qquad d = 1/11 = .090909 \qquad \Rightarrow \qquad i = d/(1 - d) = .1$

The proceeds of the future payments that are sold on 1/1/2014 are:

$\text{Proceeds} = 100\,a_{\overline{\infty}|i} = 100 \times 1/i = 1{,}000$

Note that the proceeds are based on a perpetuity -immediate since they are determined after the 1/1/2014 payment.

The semi-annual interest rate must be determined for the annuity certain. This is:

$(1 + ½i)^{½} - 1 = (1.05)^{½} - 1 = .024695$

The equation of value for the annuity certain is:

$1{,}000 \times 1.05^4 = P\,\ddot{a}_{\overline{20}|.024695} \qquad \Rightarrow \qquad P = 75.87$

Answer is B.

Question 2

The amount of the annual mortgage payment is:

$P = 100{,}000/a_{\overline{9}|} = 13{,}449$

The present value of the payments that will be made by Smith is:

$S = 13{,}449\,a_{40:\overline{9}|} = 13{,}449 \times (\ddot{a}_{40:\overline{10}|} - 1) = 13{,}449 \times 6.6923 = 90{,}004$

The mortgage will be repaid either by Smith (if still alive) or by the insurance policy (if Smith is dead). Therefore, the total present value of all payments (which is the total mortgage of \$100,000) must be equal to the sum of the present value of Smith's payments plus the present value of the death benefit.

$100{,}000 = PV_{death} + 90{,}004 \qquad \Rightarrow \qquad PV_{death} = 9{,}996$

Answer is E.

Question 3

The equation of value for the net single premium at age 70 is:

$10{,}000A_{70} = 7{,}000 \quad \Rightarrow \quad A_{70} = .7$

Using first principles,

$$A_{70} = vq_{70} + v^2{}_{1|}q_{70} + v^3{}_{2|}q_{70} + v^4{}_{3|}q_{70} + \ldots$$
$$= vq_{70} + v^2{}_{1|}q_{70} + v^3{}_{2|}q_{70} + v^3{}_{3}p_{70}A_{73}$$

Using the given values of l_x,

$q_{70} = (100 - 96)/100 = .04$
${}_{1|}q_{70} = (96 - 90)/100 = .06$
${}_{2|}q_{70} = (90 - 80)/100 = .10$
${}_{3}p_{70} = 80/100 = .80$

Substituting,

$.7 = .04v + .06v^2 + .10v^3 + .80v^3A_{73} \quad \Rightarrow \quad A_{73} = .781459$

Since the age 73 net single premium is equal to the age 70 net single premium increased by X% for 3 years,

$7{,}000 \times (1 + X)^3 = 7{,}814.59 \quad \Rightarrow \quad \ln(1 + X)^3 = \ln(7{,}814.59/7{,}000)$
$\Rightarrow \quad X = .037376$

Answer is E.

Question 4

$e_{80} = (p_{80})(1 + e_{81}) \quad \Rightarrow \quad p_{80} = .955273$
${}_{3}p_{80} = p_{80} \times {}_{2}p_{81} \quad \Rightarrow \quad {}_{2}p_{81} = .904663$

The probability that exactly two die is:

$3 \times ({}_{2}q_{81})^2 \times {}_{2}p_{81} = .024668$

The probability that all three die is:

$({}_{2}q_{81})^3 = .000867$

Total = .024668 + .000867 = .025535

Answer is D.

Question 5

The coupon for each bond is \$100 (\$5,000 × .02) every six months. The accumulated value at the end of the year of that year's coupon payment is:

$$100\, s_{\overline{2}|.025} = 202.50$$

The annual effective rate of interest, i, is: $i = 1.025^2 - 1 = .050625$

The purchase price of the twenty bonds is:

$$P = (5{,}000)(v_i^{15} + v_i^{16} + \ldots + v_i^{34}) + 2{,}000\, a_{\overline{30}|.025} + 202.50\,(Da)_{\overline{19}|i}\, v_{.025}^{30}$$
$$= 86{,}209$$

The purchase price of each individual bond is \$4,310 (\$86,209/20).

Setting up an equation of value for each individual bond,

$$4{,}310 = 5{,}000\, v_{.025}^{n} + 100\, a_{\overline{n}|.025} = 5{,}000\, v_{.025}^{n} + 100\left(\frac{1 - v_{.025}^{n}}{.025}\right) = 4{,}000 + 1{,}000\, v_{.025}^{n}$$

$$\Rightarrow \quad v_{.025}^{n} = .31 \quad \Rightarrow \quad \ln(v_{.025}^{n}) = \ln(.31) \quad \Rightarrow \quad n = \ln(.31)/\ln(v_{.025})$$
$$\Rightarrow \quad n = 47.43$$

Therefore, at a 5% yield rate, compounded semiannually, the maturity date would be 47.43 interest periods from the purchase date. Since the coupon rate is less than the yield rate, any bond with a maturity date prior to 47.43 interest periods would have a higher yield rate. This would be 23 years (47.43/2, truncated) from 1/1/2000, or 1/1/2023. There are exactly 9 bonds that mature on or before 1/1/2023.

45% of the bonds mature prior to 1/1/2023.

Answer is B.

Question 6

Smith must die within 24 months (since \$500 × 24 = \$12,000), and, similarly, Brown must die within 36 months but not before Smith. We can consider 4 cases.

I. Both Smith and Brown die during 2000, and Smith dies before Brown. The probability is:

$$½ \times (q_{80})^2 = ½ \times (.0813)^2 = .003305$$

Note the multiplication by ½ to account for the fact that there is a 50% probability that Brown will die after Smith.

II. Smith dies in 2000 and Brown dies in 2001 or 2002. The probability is:

$$(q_{80})\,(p_{80})(_{2}q_{81}) = (.0813)(.9187)(.0885 + (.9115)(.0962)) = .013159$$

III. Both Smith and Brown die during 2001, and Smith dies before Brown. The probability is:

$\frac{1}{2} \times (p_{80})^2 \times (q_{81})^2 = \frac{1}{2} \times (.9187)^2 \times (.0885)^2 = .003305$

Note the multiplication by ½ to account for the fact that there is a 50% probability that Brown will die after Smith.

IV. Smith dies in 2001 and Brown dies in 2002. The probability is:

$(p_{80})^2(q_{81})(p_{81})(q_{82}) = (.9187)^2(.0885)(.9115)(.0962) = .006550$

P = .003305 + .013159 + .003305 + .006550 = .026319

Answer is C.

Question 7

The outstanding balance is equal to the difference between the accumulated loan amount and the accumulated payments.

2% of Smith's monthly 2000 salary is \$60 (\$3,000 × .02). 2% of the increase in Smith's monthly salary in 2001 is \$3 (\$3,000 × .05 × .02).

The outstanding balance as of 1/1/2002 is:

$[5{,}000 \times (1.01)^{24}] - 60\, s_{\overline{24}|.01} - 3\, s_{\overline{12}|.01} = 4{,}692$

Answer is C.

Question 8

The annual effective rate of interest can be determined using the formula:

$i = e^{\delta} - 1 = e^{.07} - 1 = .072508$

Alternatively, an approximation of the annual effective rate of interest can be determined by treating δ as if it were $i^{(m)}$ for some very large value of m. For example, assume that m is equal to 100,000. Then,

$i = (1 + .07/100{,}000)^{100{,}000} - 1 = .072508$

The formula for duration is:

$$\bar{d} = \frac{\sum_{t=1}^{n} t v^t R_t}{\sum_{t=1}^{n} v^t R_t}$$

(where t = the time at which a payment is made and R_t = amount of payment at time t)

The duration for the bond is:

$\bar{d} = [(20)(1{,}000v^{20}) + 70v + (2)(70v^2) + \ldots + (20)(70v^{20})]/[1{,}000v^{20} + 70v + 70v^2 + \ldots + 70v^{20}]$
$= [20{,}000v^{20} + 70\,(\mathrm{Ia})_{\overline{20|}}]/[1{,}000v^{20} + 70\,\mathrm{a}_{\overline{20|}}]$
$= 11.220739$

Note that this is the MacCaulay duration. The modified duration can be determined as follows:

$y = \bar{d}/(1+i) = 11.220739/1.072508 = 10.46$

Answer is A.

Question 9

Premium $= 100{,}000\,\mathrm{A}^{1}_{35:\overline{30|}} / \ddot{\mathrm{a}}_{30:\overline{35}}$
$= 100{,}000\,(M_{35} - M_{65})/(N_{35} - N_{65})$
$= 414.57$

Note that $M_x = vN_x - N_{x+1}$

So, $M_{35} = vN_{35} - N_{36} = 85{,}287$ and $M_{65} = vN_{65} - N_{66} = 37{,}625$

Answer is A.

Question 10

The present value of the first 10 years' payments is:

$300\,\ddot{\mathrm{a}}_{\overline{10|}.07} + 300\,(\mathrm{Ia})_{\overline{9|}.07} = 11{,}152$

The payment due on 1/1/2009 is \$3,000. The payment to be made on 1/1/2010 is \$2,800. The last payment to be made on 1/1/2019 will be \$1,000. So, the present value of the last 10 years' payments is:

$1{,}000\,\ddot{a}_{\overline{10|}.06}\,v^{10}_{.07} + 200\,(D\ddot{a})_{\overline{9|}.06}\,v^{10}_{.07} = 7{,}915$

The total present value is 19,067 (11,152 + 7,915).

Answer is D.

Question 11

$q_{55}^{(w)} = q_{55}^{\prime(w)}(1 - ½\, q_{55}^{\prime(d)})/(1 - ¼\, q_{55}^{\prime(w)} q_{55}^{\prime(d)}) = .040567$

$d_{55}^{(w)} = 7{,}120 \times .040567 = 288.8$

Answer is B.

Question 12

The monthly effective rate of interest is:

$i^{(12)}/12 = (1 + .06/24)^2 - 1 = .00500625$

The annual effective rate of interest is:

$i = (1 + .06/24)^{24} - 1 = .061757$

The annual payment in contract A payable at the beginning of the first year in a single payment equivalent to the 12 monthly payments is:

$20{,}000\, a_{\overline{12}|.00500625} = 232{,}369$

The present value of the payments in contract A is:

$PV = 232{,}369 + (232{,}369)(1.0275)v + \ldots + (232{,}369)(1.0275)^4 v^4$
$= 232{,}369\, \ddot{a}_{\overline{5}|j}$ (where j = 1.061757/1.0275 - 1 = .033340)
$= 1{,}089{,}253$

The annual payment in contract B payable at the beginning of the first year in a single payment equivalent to the 24 semi-monthly payments is:

$10{,}000\, a_{\overline{24}|.0025} = 232{,}660$

The present value of the payments in contract B is:

$PV = X + 232{,}660 + (232{,}660)(1.025)v + \ldots + (232{,}660)(1.025)^4 v^4$
$= X + 232{,}660\, \ddot{a}_{\overline{5}|j}$ (where j = 1.061757/1.025 - 1 = .035860)
$= X + 1{,}085{,}496$

Setting the present values of the two contracts equal,

$X + 1{,}085{,}496 = 1{,}089{,}253 \quad \Rightarrow \quad X = 3{,}757$

Answer is B.

Question 13

The initial loan payment is:

$100{,}000/a_{\overline{360|}.01} = 1{,}028.61$

The outstanding balance immediately after the 120th repayment is:

$1{,}028.61 \times a_{\overline{240|}.01} = 93{,}418$

Under the new repayment schedule, there will be 40 payments remaining.

$93{,}418/a_{\overline{40|}.01} = 1{,}028.61 + Q \qquad \Rightarrow \qquad Q = 1{,}816.49$

Answer is D.

Question 14

$q_{75} = d_{75} / l_{75} = .10$
${}_{1|}q_{75} = d_{76} / l_{75} = (d_{76} / d_{75})(d_{75} / l_{75}) = (1/1.11)(.1)$
${}_{2|}q_{75} = d_{77} / l_{75} = (d_{77} / d_{75})(d_{75} / l_{75}) = (1/1.11)^2(.1)$
${}_{3|}q_{75} = d_{78} / l_{75} = (d_{78} / d_{75})(d_{75} / l_{75}) = (1/1.11)^3(.1)$
${}_{4|}q_{75} = d_{79} / l_{75} = (d_{79} / d_{75})(d_{75} / l_{75}) = (1/1.11)^4(.1)$

Using first principles, the net single premium is:

$$\begin{aligned} PV &= 10{,}000 q_{75} v + (1.11)(10{,}000\,{}_{1|}q_{75} v^2) + \ldots (1.11^4)(10{,}000\,{}_{4|}q_{75} v^5) \\ &= 10{,}000 \times (.1v + .1v^2 + \ldots + .1v^5) \\ &= 10{,}000 \times .1 \times a_{\overline{5|}} \\ &= 4{,}100 \end{aligned}$$

Answer is B.

Question 15

$$m_x^{(T)} = m_x^{(d)} + m_x^{(w)} = .02 + .10 = .12$$

$$p_x^{(T)} = \frac{1 - \frac{1}{2} m_x^{(T)}}{1 + \frac{1}{2} m_x^{(T)}} = .886792$$

The expected number of participants is:

$10{,}000 \times .886792 = 8{,}868$

Answer is C.

Question 16

Recall that s(x) is the same as ${}_{x}p_0$.

The probability being asked for is equal to 1 minus the probability that someone age 19 does die between the ages of 51 and 64. This probability is:

$$1 - ({}_{32}p_{19} - {}_{45}p_{19}) = 1 - ({}_{51}p_0/{}_{19}p_0 - {}_{64}p_0/{}_{19}p_0) = 1 - \left(\frac{\sqrt{49}}{10} \bigg/ \frac{\sqrt{81}}{10} - \frac{\sqrt{36}}{10} \bigg/ \frac{\sqrt{81}}{10} \right) = .8889$$

Answer is C.

Question 17

The cash flow representing the dollar-weighted rate of return is:

65,000 = 50,000(1.07) + 17,000(1.035) + 55,000(1.0175) - P(1.0525) - P(1.035) - P(1.0175)

$\Rightarrow$ P = 19,986

The time-weighted rate of return is represented by:

$$1 + i = \frac{60{,}000}{50{,}000} \times \frac{45{,}000}{60{,}000 - 19{,}986} \times \frac{40{,}000}{45{,}000 + 17{,}000 - 19{,}986} \times \frac{65{,}000}{40{,}000 + 55{,}000 - 19{,}986}$$

$\Rightarrow \quad 1 + i = 1.1133 \qquad \Rightarrow \quad i = 11.33\%$

Answer is D.

Question 18

Recall the formula for the curtate life expectancy in consecutive years:

$e_x = p_x (1 + e_{x+1})$

So,

$e_{55} = p_{55} (1 + e_{56}) \quad \Rightarrow \quad p_{55} = .990991$
$e_{56} = p_{56} (1 + e_{57}) \quad \Rightarrow \quad p_{56} = .990654$
$e_{57} = p_{57} (1 + e_{58}) \quad \Rightarrow \quad p_{57} = .990291$

Using first principles, the net single premium for the life insurance policy is:

$$\begin{aligned} PV &= 10{,}000 q_{55} v + 7{,}000 {}_{1|}q_{55} v^2 + 5{,}000 {}_{2|}q_{55} v^3 \\ &= (10{,}000)(.009009)v + (7{,}000)(.990991)(.009346)v^2 \\ &\quad + (5{,}000)(.990991)(.990654)(.009709)v^3 \\ &= 179.73 \end{aligned}$$

Answer is A.

Question 19

Since the population is stationary and the average age at termination is age 35, assume that all terminations occur at age 35. If the number of new members each year is x, then there are x people at each of ages 30 through 34 each year, and x - 100 people at each of ages 35 through 59 each year (since it is assumed that 100 people terminate at age 35 each year). The sum of the people at each age is equal to the total population. So,

$5{,}000 = 5x + 25(x - 100) \qquad \Rightarrow \qquad x = 250$

Answer is C.

Question 20

This problem can be solved using the following standard approximation:

$$\mu_{40}^{(1)} \approx \frac{d_{39}^{(1)} + d_{40}^{(1)}}{2l_{40}^{(T)}} = \frac{(e^{-39} - e^{-40}) + (e^{-40} - e^{-41})}{2 \times (60/e^{40})} = \frac{e^{-39} - e^{-41}}{2 \times (60/e^{40})} = .019587, \text{ or } 1.9587\%$$

Answer is D.

The problem can also be solved without the standard approximation, using differentiation techniques, as follows.

$$l_x^{(1)} = d_x^{(1)} + d_{x+1}^{(1)} + d_{x+2}^{(1)} + \ldots = (e^{-x} - e^{-x-1}) + (e^{-x-1} - e^{-x-2}) + (e^{-x-2} - e^{-x-3}) + \ldots = e^{-x}$$

Differentiating both sides of the equation,

$$\frac{dl_x^{(1)}}{dx} = -e^{-x}$$

Using the definition of μ_x,

$$\mu_x^{(1)} = -\frac{1}{l_x^{(T)}} \times \frac{dl_x^{(1)}}{dx} = -\frac{-e^{-x}}{(100-x)e^{-x}} = \frac{1}{100-x}$$

$$\mu_{40}^{(1)} = \frac{1}{100-40} = .0167, \text{ or } 1.67\%.$$

Note that this answer is in the same answer range as the answer using the standard approximation. However, the numerical answers are different due to the approximation.

Question 21

The present value of the gross premiums is equal to the present value of the benefits plus the present value of the expenses. The equation of value is:

$G\ddot{a}_{60} = 10{,}000A_{60} + 50A_{60} + .05G\ddot{a}_{60} + .15G\ddot{a}_{60:\overline{10|}} + .55G + 10\ddot{a}_{60}$

Rewriting using commutation functions,

$G\,N_{60} = 10{,}000\,M_{60} + 50\,M_{60} + .05G\,N_{60} + .15G\,(N_{60} - N_{70}) + .55G\,D_{60} + 10\,N_{60}$

Solving for G, G = 398

Answer is B.

Note that $M_{60} = vN_{60} - N_{61} = 10{,}628$ and $D_{60} = N_{60} - N_{61} = 36{,}709$

Question 22

The equation of value for each repayment is:

$$1{,}000 = \frac{\frac{1}{2}X}{a_{\overline{10|}.07}} + (.07)(\tfrac{1}{2}X) + \frac{\frac{1}{2}X}{s_{\overline{10|}.06}}$$

1,000 = .071189X + .035X + .037934X

X = 6,939

Answer is B.

Question 23

The effective monthly interest rate is:

$i^{(12)}/12 = 1.07^{1/12} - 1 = .005654$

The equation of value is:

$$(1{,}500)(12\,\ddot{a}_{65}^{(12)}) = 12P\,\ddot{a}_{\overline{10|}}^{(12)} + 12P\,{}_{10|}\ddot{a}_{65}^{(12)} + (.75)(12P)({}_{10|}\ddot{a}_{60}^{(12)} - {}_{10|}\ddot{a}_{60:65}^{(12)})$$
$$(1{,}500)(12\,\ddot{a}_{65}^{(12)}) = 12P\,\ddot{a}_{\overline{10|}}^{(12)} + 12P\,{}_{10|}\ddot{a}_{65}^{(12)} + (.75)(12P)({}_{10|}\ddot{a}_{60}^{(12)} - {}_{10|}\ddot{a}_{68}^{(12)})$$
$$1{,}500\,\ddot{a}_{65}^{(12)} = (P/12)\,\ddot{a}_{\overline{120|}.005654} + (P)(D_{75}/D_{65})\,\ddot{a}_{75}^{(12)} + (.75P)[(D_{70}/D_{60})\,\ddot{a}_{70}^{(12)} - (D_{78}/D_{68})\,\ddot{a}_{78}^{(12)}]$$
13,104 = 7.287140P + 2.300883P + 2.314634P - 1.384934P
P = 1,246

Answer is D.

Note that $D_{75}/D_{65} = v^{10}{}_{10}p_{65}$ $\Rightarrow$ ${}_{10}p_{65} = .705391$
Also, $\ddot{a}_x^{(12)} = N_x^{(12)} / D_x$
Finally, $\ddot{a}_{70:75}^{(12)} = \ddot{a}_{78}^{12}$ using the Gompertz table since the age in the single life annuity is equal to 3 plus the older age in the joint life annuity (as given for a 5-year age difference).

Question 24

The annual effective rate of interest can be determined using the formula:

$i = e^{\delta} - 1 = e^{.08} - 1 = .083287$

Alternatively, an approximation of the annual effective rate of interest can be determined by treating δ as if it were $i^{(m)}$ for some very large value of m. For example, assume that m is equal to 100,000. Then,

$i = (1 + .08/100{,}000)^{100{,}000} - 1 = .083287$

The quarterly effective rate of interest is:

$i^{(4)}/4 = (1.083287)^{1/4} - 1 = .020201$

Each payment is:

$250{,}000/\, a_{\overline{100}|.020201} = 5{,}840.80$

The principal portion of each repayment is $5{,}840.80v^n$, where n represents the number of repayments remaining (including the repayment currently being made). The interest portion of each repayment is $(5{,}840.80)(1 - v^n)$. The principal portion will first exceed the interest portion when $v^n > ½$. So,

$\ln(v^n) = \ln(½)$
$n = 34.6579$

Therefore, the principal component first exceeds the interest component when there are 34 payments remaining. That would be in the 67th repayment.

Answer is D.

Question 25

$$
\begin{aligned}
a_{65|\overline{50:20}} &= a_{\overline{20:50}} - a_{65:\overline{20:50}} \\
&= (a_{20} + a_{50} - a_{20:50}) - (a_{20:65} + a_{50:65} - a_{20:50:65}) \\
&= 5.12
\end{aligned}
$$

Answer is B.